MATHEMATICS OF OPERATIONS RESEARCH
Articles in Advance, pp. 1–30
ISSN 0364-765X (print), ISSN 1526-5471 (online)
https://pubsonline.informs.org/journal/moor

Effort Decisions in Contests with Shared Attributes

Jiahao He,[a] Jiheng Zhang,[a] Rachel Q. Zhang[a,*]

[a] Department of Industrial Engineering and Decision Analytics, The Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong S.A.R., China

*Corresponding author

Contact: jhebe@connect.ust.hk, https://orcid.org/0000-0002-1825-9649 (JH); jiheng@ust.hk, https://orcid.org/0000-0003-3025-1495 (JZ); rzhang@ust.hk, https://orcid.org/0000-0002-0789-8488 (RQZ)

Received: September 11, 2023
Revised: August 31, 2024; February 9, 2025; March 2, 2025; April 24, 2025
Accepted: May 6, 2025
Published Online in Articles in Advance: June 25, 2025

MSC2020 Subject Classifications: Primary: 91A12, 91B32

https://doi.org/10.1287/moor.2023.0285

Abstract. Individuals and organizations often face contests that require various skills, which can be developed through time and resource investments. Consider homogeneous contestants participating in multiple contests, each with multiple *attributes* and a *reward* for the winner or shared equally in case of a tie. Contestants can invest effort, at a cost, to enhance their skills in these attributes to maximize their expected net gain. Because contests may share some attributes while having unique ones, improving one attribute can impact winning chances differently across contests. This makes deciding how to allocate effort to each attribute a complex challenge. By reformulating the problem, we combine the effects of a contestant's efforts into their expected scores in the contests, simplifying the problem from many attributes to just the number of contests. We find that with two contests, contestants generally adopt a mixed strategy unless the contests are highly random. In less random scenarios, they tend to use more varied mixed strategies. The overall randomness of the equilibrium strategy closely resembles that of the mixed strategy in deterministic contests. We extend some of our analytical results and propose simple heuristic strategies for multiple contests, and we shed light on strategies for contestants with diverse skills.

Funding: This work was supported by the Hong Kong Research Grants Council [Grants 16200821, 16500023, and T32-615/24-R].

Keywords: noncooperative • correlated contests • resource allocation

1. Introduction

Contests are common in politics, economics, and sports. In many cases, only the top performer (or performers in the event of a tie) receives the award. For example, a politician needs to secure the most votes to win a government position, and the fastest runner claims the championship, regardless of the margin of victory. Therefore, a contestant's goal is to surpass all opponents rather than simply maximize their own performance. Each contest demands specific skills that contestants can enhance through costly efforts, with better skills increasing the likelihood of winning. The actual outcome, revealed at the end of the contest, is uncertain because of various factors, including internal conditions (like a contestant's health) and external influence (such as weather or economic and political factors).

In reality, people often participate in multiple contests. For example, an administrator might compete to chair several committees, a fashion designer may create outfits for different consumer markets, an academician conducts both fundamental and applied research, and a gymnast prepares for various contests like floor, vault, uneven bars, and beam. These contests may require similar or even opposite skills. For instance, artistic training is crucial for floor exercises but not for vaults. Showmanship might help a politician in some roles but hinder them in others. Similarly, an academician needs strong communication skills to secure industry projects, whereas technical skills are essential for fundamental research. Therefore, contestants must carefully allocate their efforts across different *attributes* to maximize their total expected net gain, which is calculated as the rewards from the contests minus total effort cost.

Most literature has implicitly assumed that contests do not share any attributes, allowing contestants to focus solely on their effort into each contest. In this view, the reward is enough to define a contest, simplifying the contestant's decision-making process because there are often more attributes than contests. In this framework, a high effort in the attributes for a specific contest suggests a higher expected reward for that contest, independent of other contests. For instance, Roberson [26] characterized contestants' equilibrium decisions in a scenario with two contestants, where their choices were represented as continuous variables.

In this paper, we start with multiple homogeneous contestants participating in two contests, each characterized by a reward and an attribute vector that indicates the importance of various attributes for winning. The two contests share some attributes, meaning that investing in an attribute to enhance the winning chances in one contest may positively or negatively affect the other contest, indicating a correlation between them. We define two contests as positively correlated if the most cost-effective efforts to improve performance in one contest positively impact the other, negatively correlated if they have a negative impact, and uncorrelated if there is no effect. We first derive contestants' equilibrium effort decisions under different correlation scenarios when the contests are deterministic, where the equilibrium strategy is mixed. Following this, we explore more general settings as described below.

1.1. Main Contributions and Findings

To our knowledge, this is the first study examining how shared attributes affect contestants' equilibrium effort decisions as existing research typically assumes that contests differ only by rewards. Incorporating attributes significantly complicates the analysis and the effort decisions of contestants. The challenge arises because an equilibrium decision may involve a mixed strategy, requiring the identification of a set of *multidimensional distribution functions* and their possibly *nonlinear* supports. Because the total number of attributes is usually much greater than the number of contests, these distributions can be high dimensional. This complexity necessitates different methodologies than those used in existing literature. By reformulating the problem, we can reduce the dimensionality from the total number of attributes to just the number of contests. Our main findings from analyzing the two-contest problem with homogeneous contestants, along with various extensions, are summarized as follows.

1. As anticipated, contestants will use a *mixed strategy* to avoid being too predictable unless the contests are highly random. In less random contests, the mixed strategy becomes more varied. We also find that the overall randomness of the equilibrium strategy closely resembles that of the mixed strategy in deterministic contests as detailed below.

2. In two *deterministic contests*, contestants will generally adopt a mixed strategy, except in one very special case. The structure of the strategies depends heavily on the contest correlation.

   a. Positively correlated contests. Contestants will aim to win both contests, and there is a clear direction for their efforts. This situation is equivalent to a single-contest problem, and the unique mixed strategy has a linear support.

   b. Uncorrelated contests. Contestants can treat the contests as independent, with their effort in each contest determined by a unique marginal distribution.

   c. Negatively correlated contests. Contestants must balance their efforts to win different contests. There is a unique mixed strategy with a nonlinear support (except in a very special case).

   Additionally, we extend our finding (a) to multiple pair-wise nonnegatively correlated contests, providing a heuristic to find an equivalent contest, and (b) to multiple uncorrelated contests. We also present an ϵ-equilibrium for multiple pair-wise nonpositively correlated contests and propose an effective heuristic procedure for developing strategies in scenarios with multiple contests exhibiting general correlations.

3. With *heterogeneous contestants*, the analysis becomes significantly more complex because the correlation between two contests is specific to each contestant. For instance, two contests may be correlated for one contestant but uncorrelated for another because of differing abilities to improve attributes. We focus on the scenario where contestants differ only in their contest correlations, and both contests are positively correlated for all contestants. Our findings indicate that only those contestants with the highest or second-highest contest correlations will exert effort if contestants have full information about their opponents. However, if contestants lack complete information, every contestant has a chance of winning each contest. In this case, uncertainty about opponents' knowledge regarding her contest correlation allows her to adopt a *pure strategy*.

1.2. Literature Review

We begin by reviewing relevant literature on a single contest, known as the *contest literature*, followed by an examination of multiple contests, referred to as the *resource allocation game literature*.

1.2.1. Single Contest. In a typical contest, contestants decide how much costly effort to exert, which collectively affects their winning probabilities through established *contest success functions*. The winner receives a reward, allowing the contest to be modeled as a game where each contestant's payoff is determined by their expected reward minus the cost of their effort. Corchón and Serena [9] provides a comprehensive survey of the contest literature, discussing justifications and equilibrium analyses under various contest success functions; the design of contest success functions; and extended models that incorporate multiple rounds, information asymmetry, and group contestants instead of individuals.

Instead of relying on a contest success function, another approach models a contestant's performance or score as a random function of their effort as seen in the innovation contests and crowdsourcing literature. Terwiesch and Xu [30] examines optimal award schemes across three types of contests, each characterized by different random performance functions. They find that when the randomness is sufficiently high, a winner-takes-all scheme is optimal for ideation and trial-and-error contests, but it may not be optimal for expertise-based contests. Under optimal winner-takes-all awards, contestants' equilibrium effort decisions are pure strategies and unique across all three types of contests. Ales et al. [2] establishes a necessary and sufficient condition for the winner-takes-all scheme to be optimal when all contestants are homogeneous in their abilities. Stouras et al. [29] studies optimal contest designs under an uncertain number of solvers with uncertain abilities, providing insights on how firms can screen out low-ability solutions and encourage high-quality ones. Mihm and Schlapp [22] investigates firms' feedback policies to solvers, including whether and when to provide feedback and the types of feedback (publicly or privately) that can improve contest outcomes. Although Terwiesch and Xu [30] finds that a larger solver population mitigates and sometimes outweighs the underinvestment of solvers, Körpeoğlu and Cho [17] demonstrates that high-quality solvers may increase their effort to win the contest, further justifying the growing popularity of open innovation contests.

Although the aforementioned works explicitly model contestants' effort decisions, another line of research directly models a contestant's decision as the distribution of their performance *score*, where the contestant with the highest realized score wins the contest. Instead of using a cost function, a constraint is placed on a contestant's expected score. Bell and Thomas [6] is the first to study equilibrium decisions in an investment contest under this model followed by Myerson [24], who examines a similar framework in the context of elections. Anderson [4] demonstrates that an equilibrium of a multistage contest among mutual fund managers can be characterized by an equilibrium score distribution at the final stage. Alpern and Howard [3] introduces a more general constraint on the score distribution and defines this scenario as a *distribution ranking game*. Because of the flexibility of the generalized constraint, they offer an innovative method to derive the equilibrium decision for the multiplayer silent duel game within this model. In these games, contestants can fully control the randomness of their performance, allowing them to adopt randomized strategies.

1.2.2. Multiple Contests. When multiple contests are involved, the problem fundamentally transforms. A contestant must decide how to allocate a limited amount of effort or resources across these contests to maximize their expected total number of wins. In this scenario, each contest is awarded to the contestant who exerts the highest level of effort. This scenario is known as a Colonel Blotto game, introduced by Borel [8], where two contestants compete across multiple contests or battlefields. Ahmadinejad et al. [1] develops a polynomial-time algorithm to compute an equilibrium.

If resources are infinitely divisible, Roberson [26] fully characterizes the equilibrium solution for the game with a general number of battlefields. Since then, researchers have identified equilibrium decisions for several variants of the basic Colonel Blotto game, primarily focusing on the case with two contestants. Some studies modify the budget constraints, such as Macdonell and Mastronardi [21], which allows nonlinear resource constraints in two contests, whereas Dziubiński [11], Hart [13], Hart [14], Kovenock and Roberson [19], and Kovenock and Roberson [20] allow budgets to be constrained in expectation. Others explore more complex objective functions; for instance, Thomas [31] constructs equilibrium solutions for contests with heterogeneous rewards, illustrated through U.S. presidential elections. Shubik and Weber [27] introduces complementarity among battlefields, where the goal shifts from winning as many battlefields as possible to winning a bundle of important ones. Rinott et al. [25] studies a Colonel Blotto gladiator game where resources (power) are allocated to gladiators in a team, and the surviving gladiators from two teams compete in a series of contests, with the last-standing team winning. To our knowledge, Boix-Adserà et al. [7] is the only paper that constructs efficiently sampleable symmetric equilibria for cases with multiple homogeneous contestants.

In the works mentioned above, contestants' performance functions are deterministic, meaning that the contestants with the highest effort or resources win (or share) the reward in a contest. In contrast, studies on ad auctions and crowdsourcing include randomness related to contestants' valuation, abilities, or random shocks. For example, Ashlagi et al. [5] analyzes how advertisers decide to participate and bid when competing in two ad auctions happening at the same time. In DiPalantino and Vojnovic [10], each contestant is allowed to participate in only one of multiple expertise-based contests, deriving participation levels when contestants' abilities are identical or unrelated across contests. Hu and Wang [15] considers a single organizer of two contests and compares two awarding mechanisms: a joint scheme, where the contestant with the highest combined performance from both contests takes all rewards, and a separate scheme, where the winner of a contest receives the reward for that contest. They find that the organizer prefers the separate (joint) scheme if the impact of randomness dominates (is dominated by) contestants' effort decisions. Körpeoğlu et al. [18] shows that allowing contestants to participate in multiple contests benefits organizers when uncertainty in contestants' performance is high enough and potential economies of scope exist across contests, meaning that increased effort in one contest may reduce costs in another because of shared investments. Stouras et al. [28] investigates how to design reward schemes for two competing firms facing noise-driven or ability-driven contestants, with the condition that each contestant can only enter one contest. Grossmann [12] examines whether to permit contestants to enter multiple unrelated contests and how to allocate prizes among those contests when there are two different types of contestants involved.

However, in all of the above work, contest attributes are not modeled, leaving their impact unexplored. In reality, each contest may require multiple skills or have multiple attributes, with attributes potentially shared across contests, leading to possible correlations. Thus, a contestant's decision becomes much more complex. We aim to model contest attributes and study contestants' equilibrium effort in all of the attributes.

1.3. Organization

The paper is organized as follows. Starting with two contests, we describe our model and reformulate it to reduce the problem dimension in Section 2. We study the impact of contest correlation because of shared attributes on contestants' equilibrium decisions for deterministic contests in Section 3 and random contests in Section 4. We shed some light on the impact of contest correlations with heterogeneous contestants and when there are more than two contests in Section 5. The paper concludes in Section 6, and all of the proofs in the paper can be found in Appendix A.

2. Model Description: Two Contests

2.1. Notation and Assumptions

Consider N homogeneous contestants, indexed by $n \in \mathcal{N} = \{1, \dots, N\}$, competing in two contests, indexed by $j \in \{1, 2\}$. These contests are characterized by a total of $M \geq 2$ attributes, with some attributes shared between both contests and others unique to each contest.

2.1.1. Contests. We characterize contest j by its reward $u_j > 0$ and an attribute vector $\boldsymbol{w}_j \in \mathbb{R}^M$, which measures the importance of the M attributes for the winning chance of contest j. A higher (lower) positive (negative) element in $\boldsymbol{w}_j$ indicates that the corresponding attribute is more desirable (detrimental) for winning contest j. A zero element signifies that the attribute is irrelevant to winning contest j. Without loss of generality, we restrict $\boldsymbol{w}_j$ to be a unit vector, allowing us to define $\boldsymbol{w}_1^T \boldsymbol{w}_2$ as the cosine of the angle between $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. Therefore, $\boldsymbol{w}_1^T \boldsymbol{w}_2 = 1$ if and only if $\boldsymbol{w}_1 = \boldsymbol{w}_2$ (indicating identical contests), and $\boldsymbol{w}_1^T \boldsymbol{w}_2 = -1$ if and only if $\boldsymbol{w}_1 = -\boldsymbol{w}_2$ (indicating opposite contests).

2.1.2. Contestants' Decisions and Performance. We define contestant n's decision as $\boldsymbol{x}_n \in \mathbb{R}^M$, where the elements represent her levels in the attributes. A positive (negative) element in $\boldsymbol{x}_n$ indicates an effort to increase (decrease; e.g., lose weight) the level of the corresponding attribute. For convenience, we will refer to $\boldsymbol{x}_n$ as contestant n's effort level. Contestant n's score in contest j is given by $\boldsymbol{w}_j^T \boldsymbol{x}_n + \xi_{nj}$, where ξ_{nj} is random noise with zero mean. Any effort directed toward an attribute contributes stochastically positively (negatively) to her score in contest j if that attribute is desired (undesired) by the contest. The contestants with the highest realized score win contest j and share the reward u_j evenly if there are multiple winners. Therefore, restricting $\boldsymbol{w}_j$ to be a unit vector is indeed without loss of generality.

2.1.3. Contestants' Costs. Contestant n incurs a quadratic cost $\boldsymbol{x}_n^T \boldsymbol{D} \boldsymbol{x}_n > 0$ for exerting effort $\boldsymbol{x}_n \neq \boldsymbol{0}$, where $\boldsymbol{D} \in \mathbb{R}^{M \times M}$ is a symmetric and positive definite matrix. A diagonal element in $\boldsymbol{D}$ indicates the cost of improving the corresponding attribute, whereas a nonzero off-diagonal element accounts for additional costs or savings when efforts are made on a pair of attributes simultaneously, reflecting their similarity or dissimilarity. Thus, $\boldsymbol{D}$ provides a comprehensive description of contestants' capabilities across the attributes.

Moreover, it can be observed that $\boldsymbol{D}^{-1} \boldsymbol{w}_j$ represents the direction along which a contestant can most efficiently improve her expected score in contest j. Consequently, $\boldsymbol{w}_j^T \boldsymbol{D}^{-1} \boldsymbol{w}_j$ measures the overall competitiveness of contestants in contest j; a higher value indicates stronger performance potential in that contest.

2.1.4. Contest Correlation Because of Shared Attributes. When a contestant improves her expected score in contest 1 along the most cost-effective direction $\boldsymbol{D}^{-1} \boldsymbol{w}_1$, the impact on her expected score in contest 2 depends on the angle between $\boldsymbol{D}^{-1}\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. We define the correlation between the two contests as $\boldsymbol{w}_1^T\boldsymbol{D}^{-1}\boldsymbol{w}_2$. The two contests are positively correlated if $\boldsymbol{w}_1^T\boldsymbol{D}^{-1}\boldsymbol{w}_2 > 0$, negatively correlated if $\boldsymbol{w}_1^T\boldsymbol{D}^{-1}\boldsymbol{w}_2 < 0$, and uncorrelated if $\boldsymbol{w}_1^T\boldsymbol{D}^{-1}\boldsymbol{w}_2 = 0$. In the uncorrelated case, improving the winning chance in contest j along $\boldsymbol{D}^{-1}\boldsymbol{w}_j$ will have no effect on the other contest. Two contests that share some common attributes can still be uncorrelated. This occurs, for instance, when $\boldsymbol{D}^{-1}\boldsymbol{w}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\boldsymbol{w}_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

2.2. Problem Formulation

Throughout the paper, we use the superscript "*" to denote functions or values in equilibrium and boldfaced letters to represent vectors or matrices, with dimensions clear from the context. With homogeneous contestants, we will focus on symmetric strategies where contestants equally share the total rewards $u_1 + u_2$ in expectation. However, they may exert different efforts and achieve different scores in the same contest when adopting a mixed strategy. To accommodate mixed strategies, we define an *equilibrium* strategy as an M-dimensional distribution $F^*(\boldsymbol{x}_n)$ that is a solution to

$$\max_F \left\{ E\left[\sum_{j=1}^{2} \frac{u_j \mathbb{1}_{\{\boldsymbol{w}_j^T\boldsymbol{x}_n + \xi_{nj} \ge \boldsymbol{w}_j^T\boldsymbol{x}_{n'}^* + \xi_{n'j},\, n' \neq n\}}}{1 + \sum_{n' \neq n} \mathbb{1}_{\{\boldsymbol{w}_j^T\boldsymbol{x}_n + \xi_{nj} = \boldsymbol{w}_j^T\boldsymbol{x}_{n'}^* + \xi_{n'j}\}}} - \boldsymbol{x}_n^T\boldsymbol{D}\boldsymbol{x}_n \right] \right\}, \tag{1}$$

where $\boldsymbol{x}_{n'}^* \sim F^*$ for $n' \neq n$. Contestant n wins contest j when $\boldsymbol{w}_j^T\boldsymbol{x}_n + \xi_{nj} \ge \boldsymbol{w}_j^T\boldsymbol{x}_{n'}^* + \xi_{n'j}$ for all $n' \neq n$, sharing the reward u_j evenly with other contestants with the same score. Problem (1) is intractable in general as it requires identifying an M-dimensional support and a distribution function.

2.3. Problem Reformulation and Dimension Reduction

In this section, we will reformulate Problem (1) as one with $\boldsymbol{D} = \boldsymbol{I}$ in Section 2.3.1 and then, reduce the dimension of the problem from the number of attributes M to the number of contests in Section 2.3.2.

2.3.1. Problem Reformulation.

Let $\tilde{\boldsymbol{x}}_n^* = \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{x}_n^*$, $\tilde{\boldsymbol{w}}_j = \frac{\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{w}_j}{\|\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{w}_j\|}$, and $\tilde{\xi}_{nj} = \frac{\xi_{nj}}{\|\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{w}_j\|}$. Then, an equilibrium solution F^* can be derived from the distribution of $\tilde{\boldsymbol{x}}_n^*$, denoted as $\tilde{F}^*$, which solves the following problem:

$$\max_{\tilde{F}} \left\{ E\left[\sum_{j=1}^{2} \frac{u_j \mathbb{1}_{\{\tilde{\boldsymbol{w}}_j^T\tilde{\boldsymbol{x}}_n + \tilde{\xi}_{nj} \ge \tilde{\boldsymbol{w}}_j^T\tilde{\boldsymbol{x}}_{n'}^* + \tilde{\xi}_{n'j},\, n' \neq n\}}}{1 + \sum_{n' \neq n} \mathbb{1}_{\{\tilde{\boldsymbol{w}}_j^T\tilde{\boldsymbol{x}}_n + \tilde{\xi}_{nj} = \tilde{\boldsymbol{w}}_j^T\tilde{\boldsymbol{x}}_{n'}^* + \tilde{\xi}_{n'j}\}}} - \tilde{\boldsymbol{x}}_n^T\tilde{\boldsymbol{x}}_n \right] \right\}, \tag{2}$$

where $\tilde{\boldsymbol{x}}_{n'}^* \sim \tilde{F}^*$ for $n' \neq n$. Problem (2) is a two-contest problem with attribute vectors $\tilde{\boldsymbol{w}}_1$ and $\tilde{\boldsymbol{w}}_2$; a cost matrix $\boldsymbol{I}$; and noises $\tilde{\xi}_{nj}$, $n = 1, \ldots, N$, $j = 1, 2$. The correlation between $\tilde{\boldsymbol{w}}_1$ and $\tilde{\boldsymbol{w}}_2$ is identical to that between $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, and the effort decisions $(\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_N^*) = \boldsymbol{D}^{-\frac{1}{2}}(\tilde{\boldsymbol{x}}_1^*, \ldots, \tilde{\boldsymbol{x}}_N^*)$. The winners under both problems are also the same. This means that we only need to analyze contestants' strategies under the cost matrix $\boldsymbol{I}$, where the correlation between the two contests is simply $\boldsymbol{w}_1^T\boldsymbol{w}_2$, and we can incorporate $\boldsymbol{D}$ later. For ease of presentation and without loss of generality, we will assume $\boldsymbol{D} = \boldsymbol{I}$ and consider general cost matrices when introducing heterogeneous contestants in Section 5.1.

2.3.2. Dimension Reduction.

Let $z_{nj} = \boldsymbol{w}_j^T\boldsymbol{x}_n$ represent contestant n's expected score in contest j given effort $\boldsymbol{x}_n$ and $\boldsymbol{W} = (\boldsymbol{w}_1, \boldsymbol{w}_2)$. An equilibrium solution can then be expressed as $F^*(\boldsymbol{x}_n) = G^*(\boldsymbol{W}^T\boldsymbol{x}_n)$, where $G^*(\boldsymbol{z}_n)$ is a two-dimensional distribution on the image of $\boldsymbol{W}^T$, denoted as $\mathrm{Im}(\boldsymbol{W}^T) = \{\boldsymbol{W}^T\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^M\}$, and solves the following optimization problem:

$$\max_G \left\{ E\left[\sum_{j=1}^{2} \frac{u_j \mathbb{1}_{\{z_{nj} + \xi_{nj} \ge z_{n'j}^* + \xi_{n'j},\, n' \neq n\}}}{1 + \sum_{n' \neq n} \mathbb{1}_{\{z_{nj} + \xi_{nj} = z_{n'j}^* + \xi_{n'j}\}}} - \gamma(\boldsymbol{z}_n) \right] \right\}, \tag{3}$$

where $\boldsymbol{z}_{n'}^* \sim G^*$ for all $n' \neq n$ and $\gamma(\boldsymbol{z})$:

$$\gamma(\boldsymbol{z}) = \min_{\boldsymbol{x}} \{\boldsymbol{x}^T\boldsymbol{x} : \boldsymbol{W}^T\boldsymbol{x} = \boldsymbol{z}\} = \begin{cases} \dfrac{z_1^2 + z_2^2 - 2\boldsymbol{w}_1^T\boldsymbol{w}_2 z_1 z_2}{1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2}, & \text{if } -1 < \boldsymbol{w}_1^T\boldsymbol{w}_2 < 1, \\ z_1^2, & \text{if } \boldsymbol{w}_1^T\boldsymbol{w}_2 \pm 1, \end{cases}$$

representing the minimum total cost for a contestant to achieve a score vector $\boldsymbol{z}$. This is because given a feasible distribution G, contestant n's expected rewards are fully determined, whereas her expected costs depend on her chosen effort $\boldsymbol{x}_t$. Therefore, contestant n will select the effort $\boldsymbol{x}_n$ that incurs the lowest cost while still achieving the desired score vector $\boldsymbol{z}_n$ in order to maximize her expected utility. Thus, we also call G^* an *equilibrium*, where contest attributes influence contestants' equilibrium strategies via the contest correlation $\boldsymbol{w}_1^T\boldsymbol{w}_2$.

A $\boldsymbol{z}_n^*$ drawn from an equilibrium G^* corresponds to contestant n's equilibrium effort given by

$$\boldsymbol{x}_n^* = \arg\min_{\boldsymbol{x}}\{\boldsymbol{x}^T\boldsymbol{x} : W^T\boldsymbol{x} = \boldsymbol{z}^*\} = \begin{cases} \dfrac{z_{n1}^* - \boldsymbol{w}_1^T\boldsymbol{w}_2 z_{n2}^*}{1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2}\boldsymbol{w}_1 + \dfrac{z_{n2}^* - \boldsymbol{w}_1^T\boldsymbol{w}_2 z_{n1}^*}{1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2}\boldsymbol{w}_2, & \text{if } -1 < \boldsymbol{w}_1^T\boldsymbol{w}_2 < 1, \\ z_{n1}^*\boldsymbol{w}_1, & \text{if } \boldsymbol{w}_1^T\boldsymbol{w}_2 \pm 1, \end{cases}$$

which is a weighted average of the attribute vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$.

Problem (3) combines the effects of a contestant's efforts on various attributes into their expected scores in the contests, thus reducing the problem's dimensionality from M to two and potentially making it more manageable. Notably, this dimensional reduction applies to any number of contests, even though $\gamma(\boldsymbol{z})$ may be an implicit function.

3. Deterministic Contests

In this section, we derive an equilibrium strategy G^* based on different contest correlations $\boldsymbol{w}_1^T\boldsymbol{w}_2$. We begin by removing randomness by setting $\xi_{nj} = 0$ for all $n \in \mathcal{N}$ and $j = 1, 2$. In this case, we can simplify our notation by dropping the subscript n. We derive equilibrium solutions $G^*(\boldsymbol{z}^*)$ when the contests are positively correlated in Section 3.1, uncorrelated in Section 3.2, and negatively uncorrelated in Section 3.3. As one can see, contestants will generally adopt mixed strategies to avoid predictability, except in one special case. The structure of these strategies is significantly influenced by the contest correlation $\boldsymbol{w}_1^T\boldsymbol{w}_2$.

3.1. Positive Correlation

When the two contests are positively correlated, there should be a direction between $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ along which contestants can maximize their net gain from both contests in equilibrium. Consequently, the problem can be reduced to one involving a single contest, for which a unique mixed-strategy equilibrium exists.

Proposition 1. *When the two contests are positively correlated, there exists a unique mixed-strategy equilibrium where the contestants sample the score in contest* 1, z_1^*, *from the distribution*

1. $G_1^*(z_1^*) = \left(\frac{z_1^{*2}}{u_1+u_2}\right)^{\frac{1}{N-1}}$ *on* $[0, \sqrt{u_1+u_2}]$, *set* $z_2^* = z_1^*$, *and exert effort* $\boldsymbol{x}^* = z_1^*\boldsymbol{w}_1$ *if* $\boldsymbol{w}_1 = \boldsymbol{w}_2$ *and*
2. $G_1^*(z_1^*) = \left(\frac{(1-\boldsymbol{w}_1^T\boldsymbol{w}_2 v)z_1^{*2}}{u_1(1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)}\right)^{\frac{1}{N-1}}$ *on* $\left[0, \sqrt{\frac{u_1(1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)}{(1-\boldsymbol{w}_1^T\boldsymbol{w}_2 v)}}\right]$, *set* $z_2^* = vz_1^*$, *where*

$$v = \frac{\sqrt{[(u_2-u_1)\boldsymbol{w}_1^T\boldsymbol{w}_2]^2 + 4u_1u_2} - (u_2-u_1)\boldsymbol{w}_1^T\boldsymbol{w}_2}{2u_1} > 0, \tag{4}$$

and exert effort $\boldsymbol{x}^* = z_1^*\left[\frac{1-\boldsymbol{w}_1^T\boldsymbol{w}_2 v}{1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2}\boldsymbol{w}_1 + \frac{v-\boldsymbol{w}_1^T\boldsymbol{w}_2}{1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2}\boldsymbol{w}_2\right]$ *if* $0 < \boldsymbol{w}_1^T\boldsymbol{w}_2 < 1$.

The problem can be simplified to a single-contest scenario with a reward of $u_1 + u_2$ and an attribute vector $\frac{\tilde{\boldsymbol{w}}}{\|\tilde{\boldsymbol{w}}\|}$, where

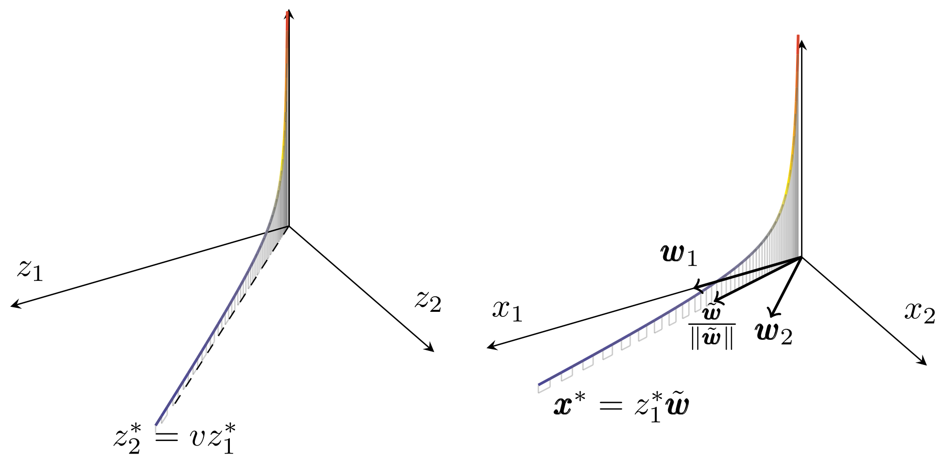
$$\tilde{\boldsymbol{w}} = \begin{cases} \dfrac{1-\boldsymbol{w}_1^T\boldsymbol{w}_2 v}{1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2}\boldsymbol{w}_1 + \dfrac{v-\boldsymbol{w}_1^T\boldsymbol{w}_2}{1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2}\boldsymbol{w}_2, & \text{if } 0 < \boldsymbol{w}_1^T\boldsymbol{w}_2 < 1, \\ \boldsymbol{w}_1, & \text{if } \boldsymbol{w}_1^T\boldsymbol{w}_2 = 1. \end{cases}$$

This attribute vector indeed lies between $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ and points in the same direction as $\boldsymbol{x}^*$. Specifically, $\tilde{\boldsymbol{w}}$ aligns with $\boldsymbol{w}_1$ when the two contests are identical and shifts closer to $\boldsymbol{w}_2$ as $u_2 - u_1$ increases. Figure 1 illustrates the density functions of $\boldsymbol{z}^*$ and $\boldsymbol{x}^*$, which have supports that are line segments. The unique equilibrium resembles that found in Alpern and Howard [3] for a constrained single-contest problem.

3.2. Uncorrelated Contests

In this scenario, $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are orthogonal, allowing contestants to treat the two contests as independent single-contest contests. Although there are infinitely many equilibrium solutions $G^*(\mathbf{z})$, contestants only need to sample

Figure 1. (Color online) The supports and density functions of $\boldsymbol{z}^*$ and $\boldsymbol{x}^*$ when $u_1 = 2u_2$ and $\boldsymbol{w}_1^T\boldsymbol{w}_2 = 0.7$.



z_j^* from the unique marginal distribution $G_j^*(z_j^*)$ as stated in Proposition 2. They allocate effort based on $z_j^*\boldsymbol{w}_j$ for contest j, leading to a total effort of $\boldsymbol{x}^* = z_1^*\boldsymbol{w}_1 + z_2^*\boldsymbol{w}_2$.

Proposition 2. *When the two contests are uncorrelated, any mixed strategy defined by a two-dimensional distribution* $G^*(\boldsymbol{z})$ *with marginal distributions* $G_j^*(z_j^*) = \left(\frac{z_j^{*2}}{u_j}\right)^{\frac{1}{N-1}}$ *on* $z_j^* \in [0, \sqrt{u_j}]$, $j = 1, 2$, *is an equilibrium.*

Proposition 2 aligns with existing results, such as Kovenock and Roberson [20, theorem 1], which characterizes equilibrium resource allocation between two heterogeneous contestants participating in multiple contests without shared attributes.

Interestingly, as N increases, competition becomes more intense, reducing the chances of winning any contest. The marginal density functions transition from increasing linear (at $N = 2$) to uniform (at $N = 3$) and then, to decreasing convex (when $N > 3$). This indicates that contestants will aim lower in each contest, exerting stochastically less effort.

3.3. Negative Correlation

When the two contests are negatively correlated, a high score in one contest corresponds to a low score in the other. Therefore, z_2^* decreases as z_1^* increases, and in equilibrium, we have $G_1^*(z_1^*) + G_2^*(z_2^*) = 1$. However, the relationship between z_1^* and z_2^* is generally complex and nonlinear, meaning that equilibrium strategies can have intricate nonlinear supports, making it challenging to determine the optimal direction for exerting efforts. We begin by examining two opposite contests, where $\boldsymbol{w}_1 = -\boldsymbol{w}_2$, in Section 3.3.1 and explore more general cases in Section 3.3.2.

3.3.1. Opposite Contests. When the two contests are exact opposites, we have $z_2^* = -z_1^*$, and the support of $\boldsymbol{z}^*$ is still a line segment as indicated in Proposition 3. Contestants will concentrate their efforts on the contest with the higher reward because winning both contests simultaneously is challenging.

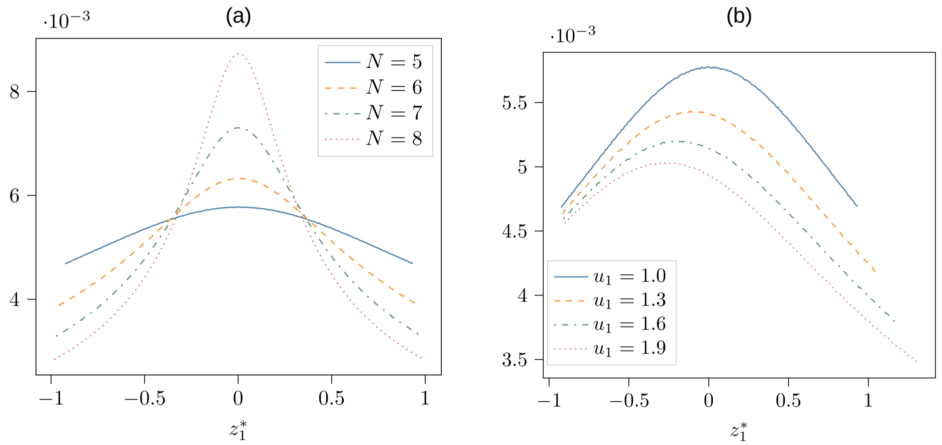
Proposition 3. *Assuming* $u_1 \geq u_2$, *when the contests are opposites, there exists a unique equilibrium where* $z_2^* = -z_1^*$.

1. *When* $N = 2$ *and* $u_1 = u_2$, *the contestants will adopt a pure strategy by exerting zero effort, ensuring that they share the total reward* $u_1 + u_2$ *evenly.*
2. *Otherwise, the contestants will adopt a mixed strategy and sample* z_1^* *from the following.*
   a. *If* $N = 2$ *and* $u_1 > u_2$, $G_1^*(z_1^*) = \frac{z_1^{*2}}{u_1 - u_2}$ *on* $[0, \sqrt{u_1 - u_2}]$.
   b. *If* $N > 3$, *the solution satisfies* $u_1[G_1^*(z_1^*)]^{N-1} + u_2[1 - G_1^*(z_1^*)]^{N-1} = z_1^2 + \hat{u}$, *on* $[-\sqrt{u_2 - \hat{u}}, \sqrt{u_1 - \hat{u}}]$, *where*

$$\hat{u} = u_1\left(\frac{\sqrt[N-2]{u_2}}{\sqrt[N-2]{u_1} + \sqrt[N-2]{u_2}}\right)^{N-1} + u_2\left(\frac{\sqrt[N-2]{u_1}}{\sqrt[N-2]{u_1} + \sqrt[N-2]{u_2}}\right)^{N-1}.$$

When $u_1 = u_2$, contestants are indifferent about winning either contest as reflected in the symmetric marginal density functions of z_1^* shown in Figure 2(a). As N increases and competition intensifies, the chances of winning any contest decrease. Contestants will exert less effort, causing the density function to become more concentrated around zero. Notably, when $N = 2$, a contestant can win one contest without any effort but cannot be the sole

Figure 2. (Color online) Marginal density function under opposite contests. (a) $u_1 = u_2 = 1$. (b) $u_2 = 1, N = 5$.



winner of both contests, regardless of their effort. Consequently, no contestant has an incentive to exert any effort in equilibrium.

When $u_1 > u_2$, contestants will concentrate more on contest 1. As a result, the marginal density function of z_1^* becomes positively skewed, with the skewness increasing as $u_1 - u_2$ grows as demonstrated in Figure 2(b). In the case of $N = 2$, both contestants will aim to win contest 1, simplifying the problem to a single-contest scenario with w_1 and a reward $u_1 - u_2$.

3.3.2. Negatively Correlated Contests with $-1 < w_1^T w_2 < 0$. In this case, the support of the scores can take the form of a nonlinear curve described by $(z_1^*(t), z_2^*(t)) = (G_1^{*-1}(t), G_2^{*-1}(1-t))$, $t \in [0,1]$. We establish the uniqueness of a mixed-strategy equilibrium and provide a strategy that converges to the equilibrium solution as the number of contestants N increases.

Proposition 4. *When the two contests are negatively correlated but not exact opposites, there exists a unique mixed-strategy equilibrium with the support and marginal distributions satisfying*

$$(z_1^*(t), z_2^*(t)) = (G_1^{*-1}(t), G_2^{*-1}(1-t)) = r(t)(\cos(\theta(t) - \theta_0), \cos(\theta(t) + \theta_0)), \quad t \in [0,1],$$

where $(r(t), \theta(t), \lambda)$ is the unique solution to

$$r^2(t) = u_1 t^{N-1} + u_2(1-t)^{N-1} - \frac{u_1 + u_2}{N} + \lambda,$$

$$\frac{\theta'(t)}{N-1} = \frac{u_1 t^{N-2}\cot(\theta + \theta_0) + u_2(1-t)^{N-2}\cot(\theta_0 - \theta)}{2\left(u_1 t^{N-1} + u_2(1-t)^{N-1} - \frac{u_1+u_2}{N} + \lambda\right)},$$

with $\theta(0) = -\theta(1) = \theta_0 = \arcsin\left(\sqrt{\frac{1 - w_1^T w_2}{2}}\right)$.

To illustrate the equilibrium solution, we consider a special case where $u_1 = u_2 = u$ and $N = 2$. In this scenario, the marginal distribution has a closed-form

$$G_1^*(z_1^*) = \frac{1}{2} + \frac{\sin\left(2\left[\theta_0 - \arccos\left(\frac{z_1^*}{r^*}\right)\right]\right) - 2w_1^T w_2\left[\theta_0 - \arccos\left(\frac{z_1^*}{r^*}\right)\right]}{2[\sin(2\theta_0) - 2w_1^T w_2 \theta_0]},$$

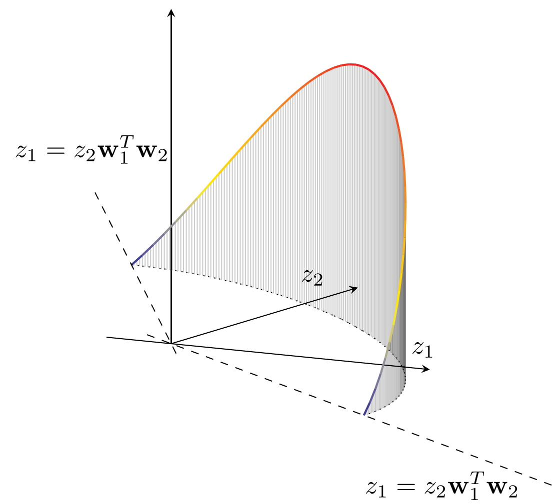
and the support of z^* forms a segment of an ellipse defined by

$$\{(z_1^*, z_2^*) : z_1^{*2} + z_2^{*2} - 2z_1^* z_2^* w_1^T w_2 = r^*[1 - (w_1^T w_2)^2], z_1^* \geq w_1^T w_2 z_2^*, z_2^* \geq w_1^T w_2 z_1^*\},$$

where $r^* = \frac{u \sin(2\theta_0)}{\sin(2\theta_0) - 2w_1^T w_2 \theta_0}$. Figure 3 illustrates the density function of z^* when $w_1^T w_2 = -0.7$.

Recall that Proposition 3 states that when the contests are exact opposites ($w_1^T w_2 = -1$), neither contestant would exert any effort as no one can be the sole winner in both contests, whereas each contestant is guaranteed to win one contest without any effort. When $-1 < w_1^T w_2 < 0$, there is a positive probability for a contestant to be the sole winner in both contests. Consequently, both contestants will adopt a mixed strategy and exert effort

Figure 3. (Color online) The density of z^* on its support when $u_1 = u_2$, $N = 2$, and $w_1^T w_2 = -0.7$.



almost surely. The density of z^* is unimodal, peaking toward the center of the support, which indicates that the contestants are more likely to aim at winning both contests. As $w_1^T w_2$ decreases, both contestants tend to aim for lower scores with reduced effort as illustrated in Figure 4.

Figure 5 illustrates the support of z^* for various values of N when $u_1 = 2u_2$ and $w_1{}^T w_2 = \frac{-1}{\sqrt{5}}$. All supports lie above the lines $z_1 = z_2 w_1^T w_2$ and $z_2 = z_1 w_1^T w_2$. As N increases, the support approaches these lines as formally established in Proposition 5.

Proposition 5. *When $-1 < w_1^T w_2 < 0$, there exist a distribution $\hat{G}$ whose support is a subset of $\{(z_1, w_1^T w_2 z_1) : z_1 \geq 0\} \cup \left\{\left(z_1, \frac{z_1}{w_1^T w_2}\right) : z_1 \leq 0\right\}$ and a random vector $(\hat{\mathbf{Z}}, \mathbf{Z}^*)$ such that $\hat{\mathbf{Z}} \sim \hat{G}$, $\mathbf{Z}^* \sim G^*$, and the norm $\|\hat{\mathbf{Z}} - \mathbf{Z}^*\| = O\left(\sqrt{N}\left(\frac{3}{4}\right)^{\frac{N}{2}}\right)$ with probability* 1.

4. Contests with Uncertainties

Suppose that contestant n's score in contest j is influenced by random noise $\xi_{nj} = \beta_j \epsilon_{nj}$, where the noise term ϵ_{nj} is a random variable that is independent and identically distributed across contestants, with zero mean, a cumultative probability distribution $H_j(\epsilon_j)$, and a probability density funtion $h_j(\epsilon_j)$. The parameter $\beta_j > 0$ scales the noise. A higher β_j indicates that the noise has a larger impact on the scores, making contest j more unpredictable.

In situations where random noise impacts outcomes, finding equilibrium strategies can be challenging even for *single-contest problems*, and there are only limited results for highly uncertain contests in the literature. For instance, studies such as those by Ales et al. [2] and Terwiesch and Xu [30] show that under certain types of noise (like Gumbel or more general distributions), a unique pure-strategy equilibrium can be established. This uniqueness of a pure strategy can be easily extended to situations with two contests if the randomness for both contests

Figure 4. (Color online) Equilibrium marginal distributions for various $w_1^T w_2$ when $u_1 = u_2 = 1$.

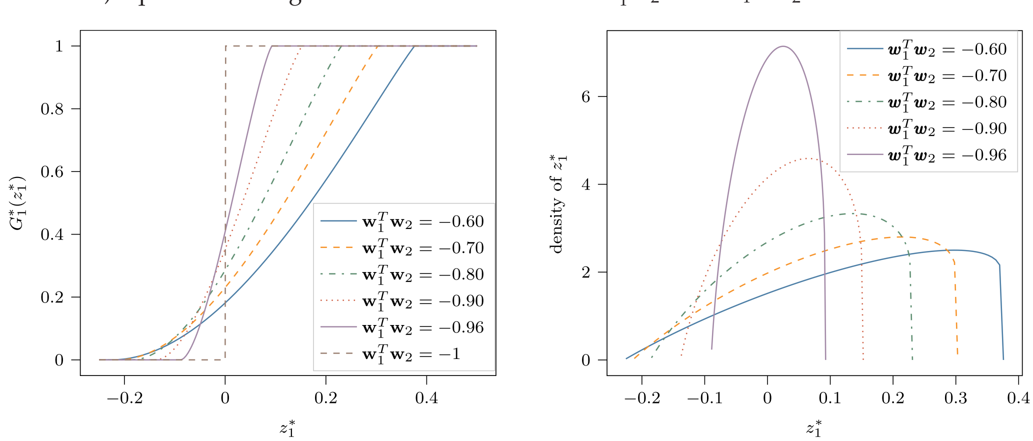
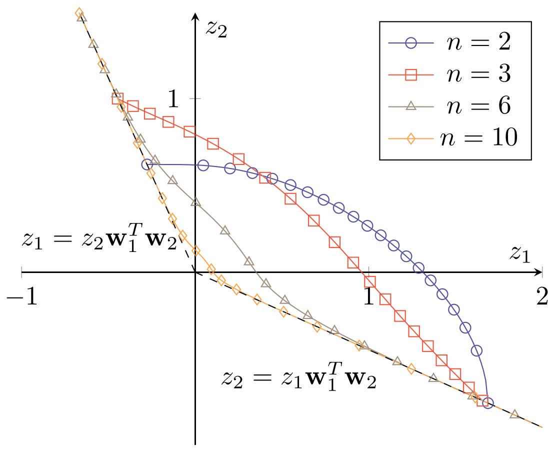


Figure 5. (Color online) Support of z^* for various N when $w_1{}^T w_2 = \frac{-1}{\sqrt{5}}$ and $u_1 = 2u_2$.



measured by β_j, $j = 1, 2$, is high enough. Because the contests are so unpredictable because of high noise, contestants do not need to introduce additional randomness into their strategies. In summary, high levels of uncertainty in contests can lead to stable strategies without the need for more randomness.

Proposition 6. *When β_1 and β_2 are high enough, the unique equilibrium is a pure strategy under which contestants exert $\boldsymbol{x}^* = \frac{N-1}{2}\left(\frac{\eta_1 u_1}{\beta_1}\boldsymbol{w}_1 + \frac{\eta_2 u_2}{\beta_2}\boldsymbol{w}_2\right)$ and achieve the expected score $\boldsymbol{z}^* = \frac{N-1}{2}\boldsymbol{W}^T\boldsymbol{W}\left(\frac{\eta_1 u_1}{\beta_1}, \frac{\eta_2 u_2}{\beta_2}\right)^T$, where $\eta_j = \int_\epsilon H_j^{N-1}(\epsilon) h_j^2(\epsilon) d\epsilon$.*

In the remaining of this section, we will focus on contests with low or moderate uncertainties. In this scenario, contestants might randomize their choices to keep their strategies unpredictable, which can complicate the analysis of equilibrium strategies. Thus, we focus on our analysis to $N = 2$ and start with a single contest. The insight derived from the single-contest problem leads to a heuristic approach for two-contest problems.

4.1. A Single Contest

With a single contest, we can simplify our notation by dropping the subscript j, and an equilibrium strategy $G^*(z^*)$ is a solution to $\max_G\{u\mathbb{P}(z + \xi_1 \geq z^* + \xi_2) - E[z^2]\}$. Following a similar argument as in the proof of Lemma A.1 in Appendix A, $G^*(z^*)$ is an equilibrium if and only if $G^*(z^*)$ minimizes

$$E[z^{*2}] + \max_z \{u\mathbb{P}(z + \xi_1 \geq z^* + \xi_2) - z^2\} = E[\hat{z}^{*2}] + \max_{\hat{z}}\{u\hat{G}^*(\hat{z}) - \hat{z}^2\} - E[(\xi_1 - \xi_2)^2],$$

where $\hat{z}^* = z^* + \xi_2 - \xi_1 \sim \hat{G}^*$. The distribution $\hat{G}^*$ is the convolution of the mixed strategy $G^*(\cdot)$ and the distribution of $\xi_2 - \xi_1$, representing contestants' winning probability based on their scores adjusted for the random noise. The convolution captures the combined effects of both contestants' strategies and the random noise that they face. With the definition of $\hat{z}^*$, we can express the problem in terms of $\hat{G}^*$ and frame it as a constrained convex optimization problem

$$\min_{\hat{G}} \int_{\hat{z}} \hat{z}^2 d\hat{G}(\hat{z}) + \max_{\hat{z}}\{u\hat{G}(\hat{z}) - \hat{z}^2\} \tag{5}$$

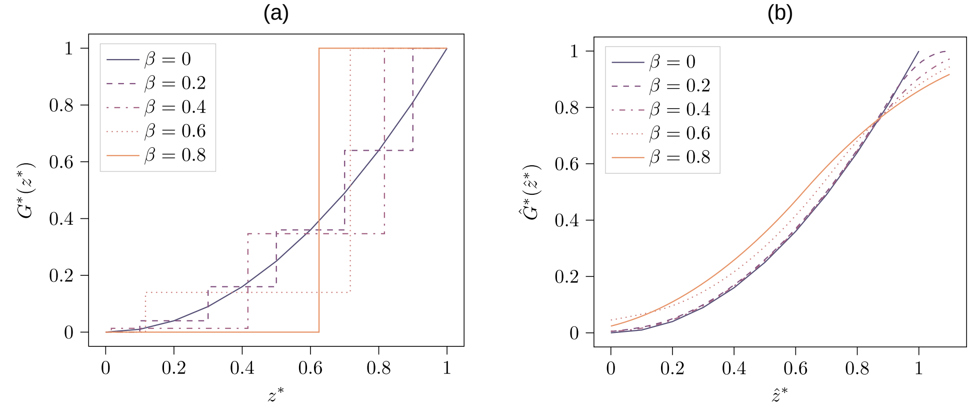
$$\text{s.t. } \hat{G}(\cdot) = G \times H\left(\frac{\cdot}{\beta}\right) \times H\left(\frac{\cdot}{-\beta}\right) \text{ for some } G. \tag{6}$$

When contests are deterministic, Constraint (6) vanishes. In this case, we have $\hat{z}^* = z^*$ and $\hat{G}^*(\hat{z}^*) = G^*(z^*) = \frac{z^{*2}}{u}$ on $[0, \sqrt{u}]$ by Proposition 1. Proposition 7 addresses the equilibrium solution to Problems (5)–(6) when the random noise ϵ_n follows a uniform distribution (a uniform shock is also used in Mihm and Schlapp [22]), which simplifies the analysis and allows for straightforward verification of the results.

Proposition 7. *Suppose that $\epsilon_n \sim U\left[-\frac{1}{2}, \frac{1}{2}\right]$.*

1. *If $\beta \geq \sqrt{\frac{u}{2}}$, the pure strategy with $z^* = \frac{u}{2\beta}$ is the unique equilibrium.*
2. *Otherwise, $\beta \in \left[\sqrt{\frac{u}{(k+1)(k+2)}}, \sqrt{\frac{u}{k(k+1)}}\right)$ for some integer $k \geq 1$, and the mixed strategy with $\mathbb{P}(z^* = z) = \frac{2\beta}{u}z$, $z \in \left\{\frac{u}{2(k+1)\beta} + (\ell - \frac{k}{2})\beta : \ell = 0, 1, \ldots, k\right\}$, is the unique equilibrium.*

Figure 6. (Color online) Equilibrium strategy and winning probability for various β under uniform noises when $u = 1$. (a) Equilibrium strategy. (b) Winning probability.



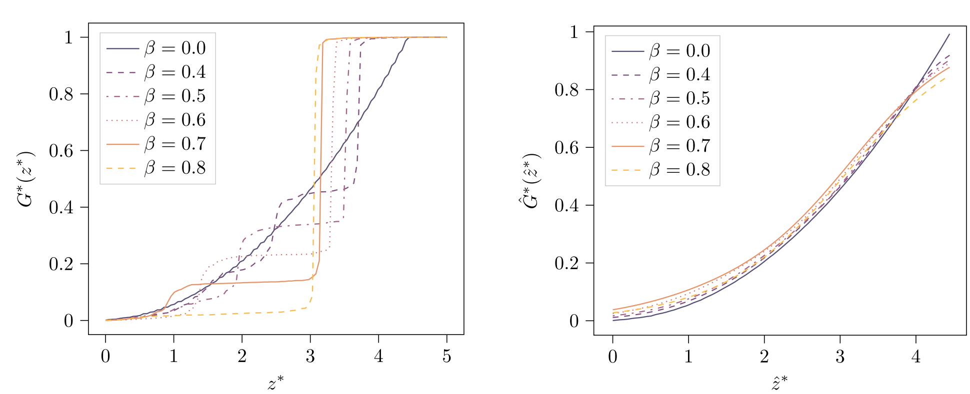
There exists a critical threshold for the scale parameter β, specifically at $\sqrt{\frac{u}{2}}$. When β is below this threshold, contestants are inclined to adopt a mixed strategy with discrete support. As β decreases (or equivalently, as k increases), the variability of the equilibrium strategy $G^*(z^*)$ increases. Higher variability in $G^*(z^*)$ indicates that contestants are diversifying their actions and potentially randomizing their choices more. This can lead to less predictability in their behavior, which is a strategic response to the uncertainties in the contest. The relationship between β and the variability of $G^*(z^*)$ can be visually represented in Figure 6(a), showing how different values of β affect the distribution of the strategies among contestants.

To understand how much randomness contestants need to add to their strategy, note that

$$\hat{G}^*(\hat{z}^*) = \begin{cases} \left[\dfrac{u}{2(k+1)\beta^2} - \dfrac{k}{2u}\right]\left[\hat{z}^* - \dfrac{u}{2(k+1)\beta} + \left(1 + \dfrac{k}{2}\right)\beta\right]^2, & \text{if } \dfrac{u}{2(k+1)\beta} - \dfrac{k\beta}{2} - \beta \le \hat{z}^* < \dfrac{u}{2(k+1)\beta} - \dfrac{k\beta}{2}, \\ \dfrac{1}{2(k+1)} - \dfrac{k\beta^2}{2u} + \dfrac{\hat{z}^{*2}}{u} - \dfrac{1}{u}\left[\dfrac{u}{2(k+1)\beta} - \dfrac{k\beta}{2}\right]^2, & \text{if } \dfrac{u}{2(k+1)\beta} - \dfrac{k\beta}{2} \le \hat{z}^* < \dfrac{u}{2(k+1)\beta} + \dfrac{k\beta}{2}, \\ 1 - \left[\dfrac{1}{2(k+1)\beta^2} + \dfrac{k}{2u}\right]\left[\dfrac{u}{2(k+1)\beta} + \dfrac{k\beta}{2} + \beta - z^*\right]^2, & \text{if } \dfrac{u}{2(k+1)\beta} + \dfrac{k\beta}{2} \le \hat{z}^* < \dfrac{u}{2(k+1)\beta} + \dfrac{k\beta}{2} + \beta, \end{cases}$$

is a *quadratic* function as illustrated in Figure 6(b) and reflects the likelihood of a contestant winning, taking into account both their mixed strategy and the randomness introduced by the contest (i.e., the noise). Furthermore, $\lim_{\beta \to 0} \hat{G}^*(\hat{z}^*) = \frac{\hat{z}^{*2}}{u}$, the equilibrium solution at $\beta = 0$. Thus, the winning probability function under the deterministic contest serves as a benchmark for mixed strategies when $\beta > 0$. Contestants can use this function to guide their strategy choices as any feasible distribution that is close to $\hat{G}^*(\hat{z}^*) = \frac{\hat{z}^{*2}}{u}$ is likely to yield a low objective value in the optimization Problems (5)–(6). Figure 7 depicts the equilibrium solution $G^*(z^*)$ for different values of

Figure 7. (Color online) Equilibrium strategy and winning probability under Gumbel noises for various β when $u = 20$.



β when the noise ϵ_n follows a Gumbel distribution with zero mean. For values of $\beta < 0.8$, contestants must introduce randomness into their decisions by adopting mixed strategies. The winning probabilities in this range remain very close to those in the deterministic scenario ($\beta = 0$). As β increases, the support of z^* becomes more concentrated. When $\beta = 0.8$, the support concentrates around a single point, indicating that the contestants' strategies are highly predictable and tightly clustered around a specific outcome. At this point, the standard deviation of the noise is approximately $0.2z^*$, showing that the randomness has a significant but controlled influence on the scores.

In summary, the analysis highlights how contestants adjust their strategies in response to the level of randomness in the contest. The deterministic equilibrium provides a foundation for mixed strategies, and as β increases, the nature of the strategies evolves from mixed to more deterministic as the support of the winning probabilities concentrates.

4.2. Two Contests

Because the problem with two contests is equivalent to two single-contest ones when they are uncorrelated, we only consider the cases where $w_1^T w_2 \neq 0$ and reformulate the problem as the following convex program:

$$\min_{\hat{G}} \int_{\hat{z}} \gamma(\hat{z}) d\hat{G}(\hat{z}) + \max_{\hat{z}} \{u_1 \hat{G}_1(\hat{z}_1) + u_2 \hat{G}_2(\hat{z}_2) - \gamma(\hat{z})\} \tag{7}$$

$$\text{s.t.} \quad \hat{G}(\cdot) = G \times H\left(\frac{\cdot}{\beta}\right) \times H\left(\frac{\cdot}{-\beta}\right) \text{ for some } G, \tag{8}$$

where $\hat{z}_j = z_j + \xi_{2j} - \xi_{1j} \sim \hat{G}_j$ and $\hat{z} \sim \hat{G}$. Again, Constraint (8) vanishes in deterministic contests, in which case equilibrium strategies have been derived in Section 3. These strategies can be used as a reference point or guide when searching for the optimal strategy $\hat{G}^*(\hat{z}^*)$ with general (β_1, β_2) numerically.

Proposition 8 extends Proposition 7 to two positively correlated contests that differ only in their attribute vectors. Contestants will adopt a pure strategy if and only if the contests exhibit high levels of randomness. Similar to Proposition 1, the problem with two positively correlated contests can be transformed into an equivalent single-contest problem with an attribute vector $\frac{w_1 + w_2}{\|w_1 + w_2\|}$ and a reward $2u$, reflecting the joint nature of the two contests.

Proposition 8. *Suppose that* $w_1^T w_2 > 0$, $u_j = u$, $\beta_j = \beta$, *and* $\epsilon_{nj} \sim U\left[-\frac{1}{2}, \frac{1}{2}\right]$, $j = 1, 2$. *Let* $\overline{\beta} = \sqrt{\frac{u(1 + w_1^T w_2)}{2}}$.

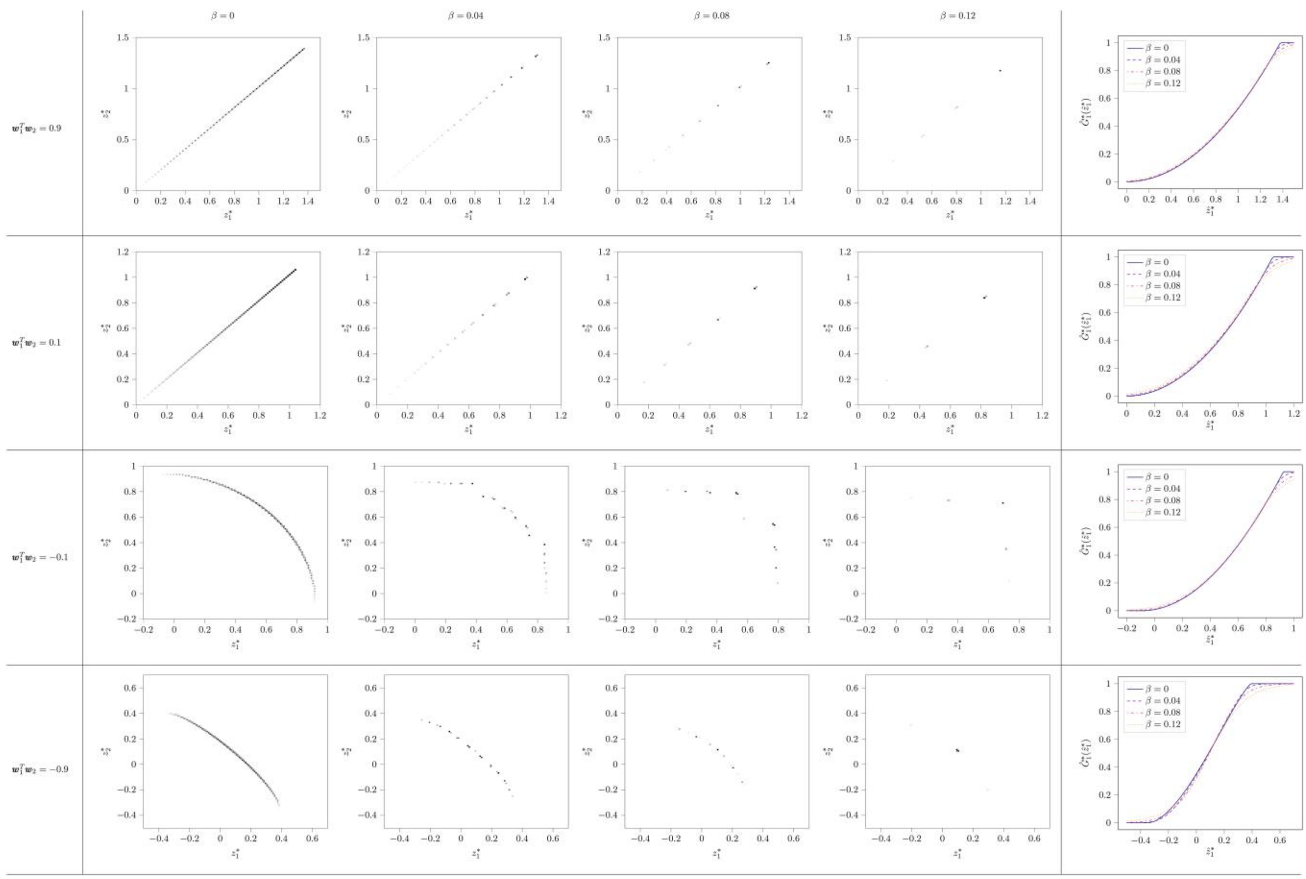
1. *If* $\beta \geq \overline{\beta}$, *there exists a unique pure strategy with* $z_1^* = z_2^* = \frac{\overline{\beta}^2}{\beta}$.
2. *Otherwise,* $\beta \in \left[\overline{\beta}\sqrt{\frac{2}{(k+1)(k+2)}}, \overline{\beta}\sqrt{\frac{2}{k(k+1)}}\right)$ *for some* $k \geq 1$, *and there exists a unique mixed strategy with* $\mathbb{P}(z^* = (z, z)) = \frac{\beta}{\overline{\beta}^2} z$, $z \in \left\{\frac{\overline{\beta}^2}{(k+1)\beta} + \left(\ell - \frac{k}{2}\right)\beta : \ell = 0, 1, \ldots, k\right\}$.

Figure 8 illustrates the support of the score z^* for various levels of correlation $w_1^T w_2$ and the scale parameter β. The shading of the plot in Figure 8 indicates the likelihood of each score being chosen, with darker shades representing higher probabilities. The analysis is conducted under the assumption that the reward $u = 1$ and that the random noise ξ_{nj} follows a Gumbel distribution with the standard deviation $\frac{\beta\pi}{\sqrt{6}}$. More details about the numerical study can be found in Appendix B.

1. For a fixed correlation (observed in each row of Figure 8), as β increases, the support of z^* becomes more limited, indicating that contestants are adding less randomness to their mixed strategies. Additionally, as β decreases, the equilibrium distributions approach those observed at $\beta = 0$. This confirms that the equilibrium strategy G^* at $\beta = 0$ serves as a useful reference for contestants when making decisions under uncertainty.
2. For a fixed β (observed in each column of Figure 8), as the contest correlation $w_1^T w_2$ decreases, the support of z^* becomes increasingly concentrated, suggesting a less random mixed strategy. High positive correlation incentivizes contestants to work harder to win both contests, minimizing reliance on luck. Conversely, a high negative correlation makes it challenging to perform well in both contests, leading contestants to depend more on luck and less on effort.
3. In the last column of Figure 8, it is observed that the winning probability $\hat{G}_1^*(\hat{z}_1^*)$ is very close to the equilibrium strategy $G_1^*(z_1^*)$ at $\beta = 0$ for small values of β. This reinforces the idea that in low randomness scenarios, contestants' strategies closely resemble those in deterministic conditions.

Figure 9 illustrates the equilibrium probability mass function when the scale parameters are set to $(\beta_1, \beta_2) = (1, 0)$. This means that contest 1 is highly random, whereas contest 2 is deterministic. Given that contest 1 resembles a "lucky draw," contestants will primarily concentrate their efforts on winning contest 2, where they can

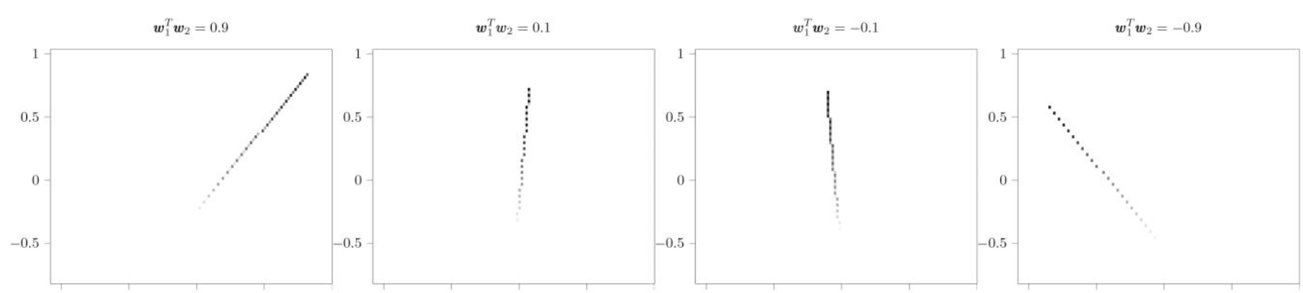
Figure 8. (Color online) Equilibrium distributions $G^*(z^*)$ for various $\boldsymbol{w}_1^T\boldsymbol{w}_2$ and $\beta_1 = \beta_2 = \beta$.



exert more control over the outcome. When the contests are highly positively correlated ($\boldsymbol{w}_1^T\boldsymbol{w}_2 = 0.9$), efforts in contest 2 positively influence the outcomes in contest 1. This means that contestants can leverage their success in the deterministic contest to improve their chances in the random contest. Conversely, when the correlation is highly negative ($\boldsymbol{w}_1^T\boldsymbol{w}_2 = -0.9$), efforts in contest 2 negatively impact contest 1. In this case, focusing on winning contest 2 may detract from performance in contest 1. For lower correlations (both positive and negative), contestants will introduce little randomness to z_1^* (the score in contest 1) and much more randomness to z_2^* (the score in contest 2). This suggests a strategic allocation of randomness based on the correlation between contests. The shapes of the support for the scores are consistent with those observed in deterministic contests; z_2^* decreases as z_1^* increases when $\boldsymbol{w}_1^T\boldsymbol{w}_2 < 0$, indicating a trade-off between the two contests, and increases in z_1^* when $\boldsymbol{w}_1^T\boldsymbol{w}_2 > 0$, suggesting a complementary relationship between the two contests.

In summary, contestants will opt for a mixed strategy to maintain unpredictability in their actions to their opponents. As the uncertainty in contests decreases, contestants tend to introduce more randomness into their mixed strategies. This means that they are less reliant on deterministic outcomes and more reliant on varied

Figure 9. Equilibrium distributions $G^*(z^*)$ under various $\boldsymbol{w}_1^T\boldsymbol{w}_2$ when $(\beta_1, \beta_2) = (1, 0)$.



actions. The overall randomness in their strategies aligns closely with the randomness implied by mixed strategies in deterministic contests. This suggests that even in uncertain environments, contestants can draw on principles from deterministic scenarios to guide their strategic choices. In essence, contestants balance their strategies between randomness and predictability based on the level of uncertainty in the contest, achieving a nuanced approach that leverages insights from both deterministic and stochastic contexts.

5. Extensions

The basic model considered homogeneous contestants participating in two contests. This setup provided a clear framework for analyzing strategies and outcomes. In this section, we explore more complex competitive environments by considering heterogeneous contestants in two deterministic contests in Section 5.1 and homogeneous contestants in multiple contests in Section 5.2, thereby broadening the scope of the analysis.

5.1. Heterogeneous Contestants in Two Deterministic Contest

Suppose that contestant n incurs a quadratic cost associated with their effort $\boldsymbol{x}_n$, $\boldsymbol{x}_n^T\boldsymbol{D}_n\boldsymbol{x}_n$, where $\boldsymbol{D}_n$ is a positive definite matrix that captures how different dimensions of contestant n's effort contribute to the overall cost. The term $\boldsymbol{w}_j^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_j$ quantifies contestant n's overall competitiveness in event j and measures how effectively contestant n can leverage her attributes relative to the costs incurred. The term $\boldsymbol{w}_1^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_2$ represents the correlation between events for contestant n. It is possible for two events to be correlated for one contestant while remaining uncorrelated for another. This nuance adds complexity to the analysis of competitive strategies and outcomes. An equilibrium solution consists of a tuple of M-dimensional distributions $(F_1^*(\boldsymbol{x}_1^*),\ldots,F_N^*(\boldsymbol{x}_N^*))$ such that $F_n^*(\boldsymbol{x}_n^*)$ is a solution to

$$\max_{F_n}\left\{E\left[\sum_{j=1}^{2}\frac{u_j 1_{\{\boldsymbol{w}_j^T\boldsymbol{x}_n+\xi_{nj}\geq \boldsymbol{w}_j^T\boldsymbol{x}_{n'}^*+\xi_{n'j},\,\forall n'\in\mathcal{N}_{-n}\}}}{1+\sum_{n'\in\mathcal{N}_{-n}}1_{\{\boldsymbol{w}_j^T\boldsymbol{x}_n+\xi_{nj}=\boldsymbol{w}_j^T\boldsymbol{x}_{n'}^*+\xi_{nj'}\}}}-\boldsymbol{x}_n^T\boldsymbol{D}_n\boldsymbol{x}_n\right]\right\}, \tag{9}$$

where $\boldsymbol{x}_{n'}^*\sim F_{n'}^*$ for all $n'\in\mathcal{N}_{-n}=\{1,\ldots,n-1,n+1\ldots,N\}$. Thus, finding an equilibrium solution requires solving n optimization problems simultaneously.

Again, letting $z_{nj}=\boldsymbol{w}_j^T\boldsymbol{z}_n$, we have that $F_n^*(\boldsymbol{x}_n^*)=G_n^*(\boldsymbol{W}^T\boldsymbol{x}_n)$, where $(G_1^*(\boldsymbol{z}_1^*),\ldots,G_N^*(\boldsymbol{z}_N^*))$ is a tuple of two-dimensional distributions such that $G_n^*(\boldsymbol{z}_n^*)$, $1\leq n\leq N$, is a solution to

$$\max_{G_n}\left\{E\left[\sum_{j=1}^{2}\frac{u_j 1_{\{z_{nj}+\xi_{nj}\geq z_{n'j}^*+\xi_{n'j},\,\forall n'\in\mathcal{N}_{-n}\}}}{1+\sum_{n'\in\mathcal{N}_{-n}}1_{\{z_{nj}+\xi_{nj}=z_{n'j}^*+\xi_{n'j}\}}}-\gamma_n(\boldsymbol{z}_n)\right]\right\} \tag{10}$$

given $\boldsymbol{z}_{n'}^*\sim G_{n'}^*$ for all $n'\in\mathcal{N}_{-n}$, where

$$\gamma_n(\boldsymbol{z})=\min_{\boldsymbol{x}}\{\boldsymbol{x}^T\boldsymbol{D}_n\boldsymbol{x}:\boldsymbol{W}^T\boldsymbol{x}=\boldsymbol{z}\}=\begin{cases}\boldsymbol{z}^T\hat{\boldsymbol{D}}_n^{-1}\boldsymbol{z}, & \text{if } -1<\boldsymbol{w}_1^T\boldsymbol{w}_2<1,\\[2mm] \dfrac{z_1^2}{\boldsymbol{w}_1^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_1}, & \text{if } \boldsymbol{w}_1^T\boldsymbol{w}_2=\pm 1,\end{cases}$$

$$\hat{\boldsymbol{D}}_n=\begin{pmatrix}\boldsymbol{w}_1^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_1 & \boldsymbol{w}_1^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_2\\ \boldsymbol{w}_1^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_2 & \boldsymbol{w}_2^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_2\end{pmatrix}.$$

The $M\times M$ cost matrix $\boldsymbol{D}_n$ influences contestant n's equilibrium decision G_n^* primarily through two metrics: overall competitiveness in the two contests $\boldsymbol{w}_j^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_j$ and contest correlation $\boldsymbol{w}_1^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_2$. They are captured by the 2×2 matrix $\hat{\boldsymbol{D}}_n$, which reduces to a scalar $\boldsymbol{w}_1^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_1$ if the contests are identical or opposites.

The introduction of heterogeneous contestants greatly complicates the analysis. To isolate the effects of contest correlation because of shared attributes, we consider two deterministic contests with identical rewards and contestants that differ only by the extent that the contests are positively correlated. That is, $\xi_{nj}=0$, $u_j=u$, $\boldsymbol{w}_j^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_j=c>0$ (i.e., contestants are equally competitive in both contests), and $\boldsymbol{w}_1^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_2>0$, in which case $z_{n1}^*=z_{n2}^*=z_n^*$ for all $1\leq n\leq N$.

5.1.1. Under Full Information. If $\boldsymbol{D}_n$ is known to all contestants, they can be grouped into different classes based on their contest correlations. Contestants with the kth-highest correlation, denoted as $\delta_k>0$, are referred to as class k contestants. Each class adopts the same strategy regardless of their cost matrix as contestants in each class share the same $\hat{\boldsymbol{D}}_n$. Let n_1 and n_2 be the numbers of class 1 and class 2 contestants, respectively.

Proposition 9. *Assume that contestants vary only in the degree of positively correlation positive correlation two contests.*

1. *If $n_1 = 1$, only contestants in classes 1 and 2 will put in effort, and*

$$G^*_{nj}(z^*_n) = \begin{cases} \dfrac{z_n^{*2}}{u(c+\delta_2)}\left[\dfrac{u(c+\delta_1)}{z_n^{*2}+u(\delta_1-\delta_2)}\right]^{\frac{n_2-1}{n_2}}, & \text{if } n \text{ is in class } 1, \\ \left[\dfrac{z_n^{*2}+u(\delta_1-\delta_2)}{u(c+\delta_1)}\right]^{\frac{1}{n_2}}, & \text{if } n \text{ is in class } 2, \end{cases}$$

on $[0, \sqrt{u(c+\delta_2)}]$.

2. *If $n_1 > 1$, only class 1 contestants will exert effort, and $G^*_{nj}(z^*_n) = \left[\frac{z_n^{*2}}{u(c+\delta_1)}\right]^{\frac{1}{n_1-1}}$ on $[0, \sqrt{u(c+\delta_1)}]$ for all n in class 1.*

Contestants with higher contest correlations perform better and are thus more competitive. Class 1 contestants will always put in effort to win both contests. Class 2 contestants will exert effort if there is a single class 1 contestant. In this scenario, the problem is equivalent to a single-contest one with a reward $2u$, an attribute vector $\frac{\boldsymbol{w}_1+\boldsymbol{w}_2}{\|\boldsymbol{w}_1+\boldsymbol{w}_2\|}$, and $\gamma_n(z,z) = \frac{2z^2}{1+\delta_k}$ for contestant n in class k, where $k = 1, 2$. If there are multiple class 1 contestants, the problem can be analyzed as having n_1 homogeneous contestants as discussed in Proposition 1. Contestants in class $k \geq 3$ will not exert any effort.

5.1.2. Under Partial Information. Assume that the contestants are fully aware of their own contest correlations but only know the distribution of their opponents' contest correlations. Specifically, these correlations are positive random variables drawn independently from a common continuous distribution Q. We further assume that $\boldsymbol{w}_j^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_j = c$ is maintained across all realizations of the correlations under the distribution Q. This condition can be easily met because the cost matrix $\boldsymbol{D}_n$ is $\frac{M(M-1)}{2}$ dimensional and typically, $M \gg 2$.

Proposition 10. *When contestants vary only by the degree of positive correlation between two contests and view their contestants' contest correlations as random variables drawn independently from a continuous distribution Q, contestant n will adopt a pure strategy given by $z^*_n = \sqrt{u\int_0^{\boldsymbol{w}_1^T\boldsymbol{D}_n^{-1}\boldsymbol{w}_2}(c+\delta)dQ^{N-1}(\delta)}$.*

All contestants will participate by exerting effort because there is a positive probability that a contestant's correlation is higher than that of others, regardless of her actual contest correlation. The higher her contest correlation, the more effort she will exert, and the higher her chances are of winning the contests. Moreover, unlike scenarios with homogeneous contestants, contestants in this case only need to adopt a pure strategy, even in deterministic contests with heterogenous contestants. This is because a contestant's pure strategy, which incorporates her private information (her true contest correlation), appears as a "mixed" strategy to her opponents, even when the contests are deterministic. In contrast, with homogeneous contestants, there is no private information, and every contestant makes her decision based solely on public information. Therefore, unless the contests are highly random, a pure strategy could lead to overly predictable decisions.

5.2. Homogeneous Contestants in Multiple Contests

Suppose there are N homogeneous contestants competing in J contests, each with a reward u_j and an attribute vector $\boldsymbol{w}_j$, $j = 1, \ldots, J$. In this case, $\boldsymbol{W} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_J)$. As in Section 4, we assume that the noise is given by $\xi_{nj} = \beta_j\epsilon_{nj}$, where $\epsilon_{1j}, \ldots, \epsilon_{Nj}$ are independently and identically distributed with zero mean. Proposition 11 below generalizes Proposition 6 to the case of J contests when the contests are highly random. In this scenario, a contestant will exert effort given by $\frac{N-1}{2}\frac{\eta_j u_j}{\beta_j}\boldsymbol{w}_j$. This effort is directed toward winning contest j and relies solely on information pertinent to that specific contest j.

Proposition 11. *When $\beta_1, \ldots, \beta_N$ are sufficiently high, the unique equilibrium is a pure strategy where contestants target the expected score given by $\boldsymbol{z}^* = \frac{N-1}{2}\boldsymbol{W}^T\boldsymbol{W}\left(\frac{\eta_1 u_1}{\beta_1}, \ldots, \frac{\eta_J u_J}{\beta_J}\right)^T$ by exerting effort $\boldsymbol{x}^* = \frac{N-1}{2}\sum_{j=1}^J \frac{\eta_j u_j}{\beta_j}\boldsymbol{w}_j$.*

In the remainder of this section, we will focus on deterministic contests. First, we note that if $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_J$ are linearly dependent, we can introduce a unique dummy attribute for each contest, creating J linearly independent contests denoted as $\tilde{\boldsymbol{w}}_1^{(k)}, \ldots, \tilde{\boldsymbol{w}}_J^{(k)}$ for some $k > 0$. Specifically, we define $\tilde{\boldsymbol{w}}_j^{(k)} = \begin{pmatrix} \boldsymbol{w}_j \\ \frac{1}{k}\boldsymbol{e}_j \end{pmatrix}$, where $\boldsymbol{e}_j \in \mathbb{R}^J$ is the vector with a one in the jth coordinate and zero elsewhere. Simultaneously, we expand the cost matrix to $\tilde{\boldsymbol{D}}^{(k)} = \begin{pmatrix} \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & k\boldsymbol{I}_J \end{pmatrix}$, where $\boldsymbol{I}_J$ is a J-dimensional identity matrix. For sufficiently large k, any efforts to improve the dummy attributes will have a negligible effect on winning the contests. Proposition 12 establishes that as $k \to +\infty$, contestants will eventually cease to make efforts toward the dummy attributes. Consequently, a sequence of equilibrium strategies, denoted as $(G_1^{k*}, \ldots, G_n^{k*})$, $k = 1, 2, \ldots$, will converge to that of the original problem. Therefore, we only need to analyze the case of linearly independent contests.

Proposition 12. *A subsequence of equilibrium solutions* $\{(G_1^{k*}, \ldots, G_n^{k*}) : k = 1, 2, \ldots\}$ *converges to an equilibrium of the original problem.*

Next, we present some analytical results for contests that are pair-wise nonnegatively (nonpositively) correlated or uncorrelated. These results will serve as a foundation for developing heuristic approaches applicable to more general scenarios.

5.2.1. Pair-Wise Nonnegatively Correlated Contests. Proposition 13 extends Proposition 1 to the case of multiple contests. It establishes that if the contests are pair-wise nonnegatively correlated, the problem can be effectively reduced to a single-contest scenario.

Proposition 13. *If* $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_J$ *are linearly independent and* $\boldsymbol{w}_j^T \boldsymbol{D}^{-1} \boldsymbol{w}_{j'} \geq 0$ *for all* j *and* j', *the problem can be reduced to a single-contest one.*

However, identifying the equivalent single contest requires solving a system of multivariate quadratic equations, which is nontrivial. Therefore, we propose a simple heuristic procedure and conduct a numerical experiment. We begin by applying Proposition 1 to the two contests with the highest correlation to find the equivalent contest, effectively merging the two into one. We continue this process until only a single contest remains.

Because contestants share the total reward $\sum_{j=1}^J u_j$ equally in expectation, we measure the effectiveness of the heuristic procedure by the potential percentage gain if a contestant deviates from the heuristic strategy. This is given by $\eta = \frac{R^*}{\frac{1}{N}\sum_{j=1}^J u_j} - 1$, where R^* is the highest expected reward that a contestant can achieve by deviating from the heuristic strategy. The smaller value of η indicates a more effective heuristic procedure.

To evaluate the efficiency of the heuristic, we generated 1,000 problem incidents with $J = 6$ and $M = 2$. In each instance, the attribute vector $\boldsymbol{w}_j$ was drawn uniformly from the set of nonnegative unit vectors in $\mathbb{R}^2$, whereas the reward u_j was drawn from the folded standard normal distribution. Figure 10(a) displays the average and percentiles of η for different numbers of contestants: $N = 12, 24, 48, 96$. The results indicate that our heuristic performs quite well, although η tends to increase with N, which is expected. Furthermore, merging contests based on their correlations proves to be significantly more effective than doing so in a random order as shown in Figure 10(b) or based on their rewards from highest to lowest as depicted in Figure 10(c). This suggests that correlations have a more substantial impact on contestants' strategy than rewards do.

5.2.2. Pair-Wise Nonpositively Correlated Contests. Proposition 14 identifies an ε-equilibrium for scenarios where all contests are pair-wise negatively correlated or uncorrelated. In this case, the support of the equilibrium is a union of J line segments given by $\{t\sqrt{\boldsymbol{D}}^{-1}\boldsymbol{w}_j | t \geq 0\}$, $j = 1, 2, \ldots, J$.

Figure 10. (Color online) Average and percentiles of potential percentage gain with nonnegatively correlated contests. (a) The heuristic. (b) Random order. (c) Order by rewards.

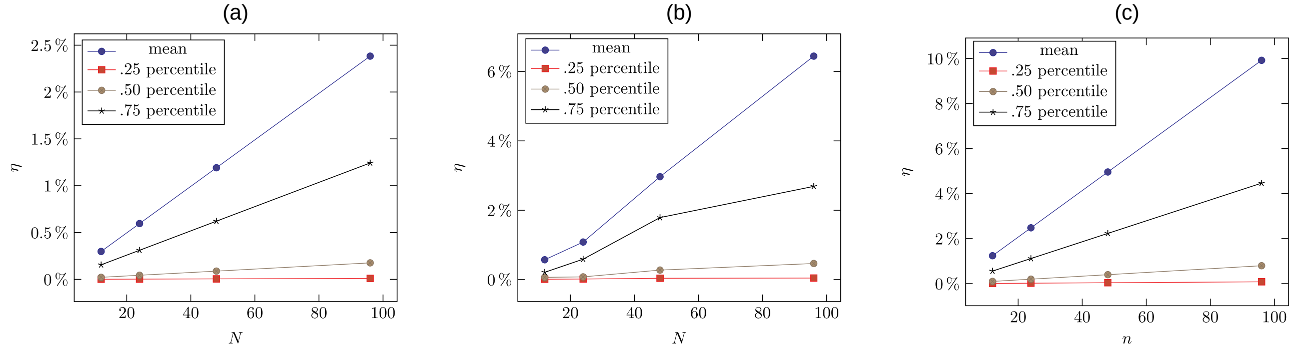
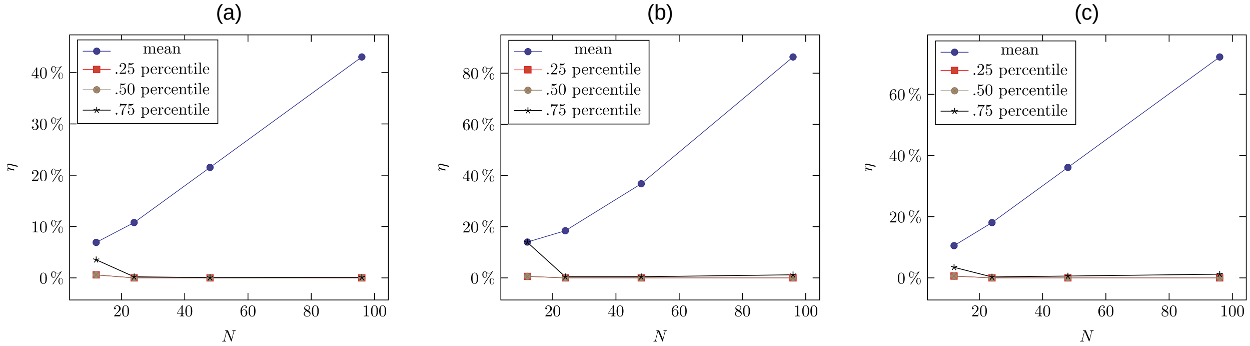


Figure 11. (Color online) Average and percentiles of potential percentage gain with contests of general correlation. (a) The heuristic. (b) Random order. (c) Order by rewards.



Proposition 14. *If $w_1, w_2, \ldots, w_J$ are linearly independent and $w_j^T D^{-1} w_{j'} \leq 0$ for all j and j', then for any given $\varepsilon \geq \left(\frac{J-1}{J}\right)^N \sum_{j=1}^J u_j$, an ε-equilibrium exists. This equilibrium assigns an equal probability mass of $\frac{1}{J}$ on a segment of the line $\{t\sqrt{D}^{-1} w_j \,|\, t \geq 0\}$, $j = 1, 2, \ldots, J$.*

This ε-equilibrium suggests that when there is a sufficient number of contestants, focusing on a randomly chosen contest—regardless of its reward—can be a viable strategy. We can derive an ε-equilibrium strategy $G(z)$ where $z = W^T x$ such that

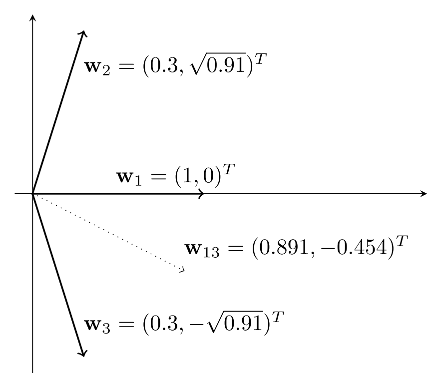
$$P(x \in \{t\sqrt{D}^{-1} w_j \,|\, t \in [0, z_j]\}) = 1 \wedge \left[\left(\frac{J-1}{J}\right)^{n-1} + \frac{z_j^2}{u_j}\right]^{\frac{1}{N-1}} - \frac{J-1}{J}$$

for all $1 \leq j \leq J$ and $z_j \geq 0$ as provided in the proof of Proposition 14.

5.2.3. General Contests. Drawing inspiration from Propositions 13 and 14, we can approach general contests by first identifying the two contests with the highest positive correlation and merging them to find an equivalent single contest. This process continues until all of the remaining contests are pair-wise nonpositively correlated, at which point we can apply the procedure outlined in Section 5.2.2.

For our numerical experiments, we generated 1,000 problem instances with $J = 3$ and $w_j \in \mathbb{R}^2$. In each instance, the rewards were sampled from the folded standard normal distribution, similar to the approach in Section 5.2.1, whereas the attribute vectors were generated uniformly from the set of all unit vectors. Figure 11 displays the average and percentiles of η for $N = 12, 24, 48, 96$. Intuitively, if we can effectively group contests into negatively correlated clusters, each containing "highly" positively correlated contests, our heuristic strategy should perform well, particularly when N is large. Conversely, if weakly positively correlated contests are merged, a significant amount of information about individual contests may be lost, which could lead to a higher average of η as shown in Figure 11. For example, consider three contests with rewards $(u_1, u_2, u_3) = (2, 0.5, 1)$ and attribute vectors (w_1, w_2, w_3) illustrated in Figure 12. In this scenario, w_1 is weakly positively correlated with both w_2 and w_3,

Figure 12. An example with three contests.



whereas $\boldsymbol{w}_2$ and $\boldsymbol{w}_3$ are negatively correlated. Suppose we merge $\boldsymbol{w}_1$ and $\boldsymbol{w}_3$ to form an equivalent single contest represented by $\boldsymbol{w}_{13} = (0.891, -0.454)$ with a reward of $u_{13} = 3$. Consequently, $\boldsymbol{w}_{13}$ becomes negatively correlated with $\boldsymbol{w}_2$, resulting in the loss of the positive correlation between $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. This can lead to a high value of $\eta = \frac{R^*}{\frac{1}{N}(u_1+u_2+u_3)} - 1$. In general, our heuristic struggles when the contests are more evenly distributed or weakly correlated. In such cases, it becomes unclear what an equilibrium solution should look like and how to derive one, reflecting many of the complex decisions that we encounter in real life.

6. Conclusions and Discussions

Motivated by a wide range of applications involving multiple contests faced by individuals and organizations, we study a class of problems characterized by three key features. First, in each contest, the reward is awarded to the winner or winners in the case of a tie, regardless of the winning margin. Thus, a contestant's objective is to outperform opponents rather than simply maximizing their individual performance. Second, contestants can enhance their probability of winning by exerting effort into the relevant attributes, although this comes at a cost. The more effort that they invest, the higher their chances of winning are, and the actual outcomes are influenced by randomness noise. Third, contests may be correlated because of shared attributes, which introduces a fundamental challenge: balancing the costs of *efforts into the attributes* with the potential rewards from possibly *correlated contests* because of shared attributes.

By reformulating the problem, we can combine the effects of a contestant's efforts into their expected scores in the contests, simplifying the problem from many attributes to just the number of contests. As expected, contestants generally adopt a mixed strategy to avoid being too predictable to their opponents unless the contests are highly random. In less random scenarios, they tend to use more varied mixed strategies. However, the strategies that contestants employ are heavily dependent on the correlations between the contests. For instance, when contests are pair-wise nonnegatively correlated, the problem with homogeneous contestants can be reduced to a single-contest scenario. Conversely, an ϵ-strategy exists when contests are pair-wise nonpositively correlated. We provide heuristic approaches for general contest correlations and extend our analysis to heterogenous contestants engaged in two contests.

Additionally, our work offers a theoretical framework that can be applied to broader problems. For example, it can assist contest organizers in making decisions regarding the rewards. Suppose the organizer of contest j, $j = 1, 2$, wants to maximize the expected highest score (performance) less the reward

$$\Pi(u_j | u_{j'}) = E[\max\{z^*_{1j}, \ldots, z^*_{Nj}\}] - u_j,$$

where $\boldsymbol{z}^*_n$ represents contestant n's equilibrium score vector given the rewards (u_1, u_2). The organizers' equilibrium solution (u^*_1, u^*_2) is designed to maximize $\Pi(u^*_j | u^*_{j'})$, $j' = 1, 2$, and $j' \neq j$. Our theoretical framework provides insights into the organizers' decisions as established by the following result.

Proposition 15. *When $\boldsymbol{D}_n = \boldsymbol{I}$, $\xi_{nj} = 0$, and $\boldsymbol{w}_1^T \boldsymbol{w}_2 > 0$, the optimal rewards are given by $u^*_1 = u^*_2 = \frac{N^2(2-\boldsymbol{w}_1^T\boldsymbol{w}_2)^2(1+\boldsymbol{w}_1^T\boldsymbol{w}_2)}{4(3N-1)^2}$.*

In other words, when the two contests are positively correlated and deterministic, the equilibrium rewards decrease with increasing contest correlation as higher correlations make contestants' efforts more effective.

Appendix A. Technical Proofs

Lemma A.1. *Suppose that $\boldsymbol{w}_1^T \boldsymbol{w}_2 \neq -1$, $\boldsymbol{D}_n = \boldsymbol{I}$, and $\xi_{nj} = 0$ for all $n = 1, \ldots, N$ and $j = 1, 2$. A distribution G^* is a symmetric equilibrium if and only if*

$$\max_{\boldsymbol{z} \in \mathrm{Im}(\boldsymbol{W}^T)} \left\{ \sum_{j=1}^{2} u_j G_j^{*N-1}(z_j) - \gamma(\boldsymbol{z}) \right\} + E_{G^*}[\gamma(\boldsymbol{z}^*)] = \frac{u_1 + u_2}{N}. \tag{A.1}$$

Furthermore, the support of G^* is a subset of $\arg\max_{\boldsymbol{z} \in \mathrm{Im}(\boldsymbol{W}^T)} \{\sum_{j=1}^{2} u_j G_j^{*N-1}(z_j) - \gamma(\boldsymbol{z})\}$, *and z^*_2 increases (decreases) in z^*_1 when $\boldsymbol{w}_1^T \boldsymbol{w}_2 > 0$ $(\boldsymbol{w}_1^T \boldsymbol{w}_2 < 0)$.*

Proof of Lemma A.1. Because the contestants are homogeneous and assumed to adopt the same strategy G^*, they will share equally the total reward $u_1 + u_2$ in expectation, and thus, the optimal objective value should be exactly $\frac{u_1+u_2}{N} - E_{G^*}[\gamma(\boldsymbol{z}^*)]$. We can also obtain the optimal objective value by finding a fixed point $\boldsymbol{z}$ (i.e., a distribution with a single point) that maximizes the objective function $E\left[\sum_{j=1}^{2}\frac{u_j 1_{\{z_j\geq z^*_{n'j},n'\neq n\}}}{1+\sum_{n'\neq n}1_{\{z_{nj}=z^*_{n'j}\}}}-\gamma(z)\right]$: that is,

$$
\begin{aligned}
&\max_{G}\left\{E\left[\sum_{j=1}^{2}\frac{u_j 1_{\{z_{nj}\geq z^*_{n'j},n'\neq n\}}}{1+\sum_{n'\neq n}1_{\{z_{nj}=z^*_{n'j}\}}}-\gamma(z_n)\right]\right\}\\
&=\max_{z\in\mathrm{Im}(W^T)}\left\{E\left[\sum_{j=1}^{2}\frac{u_j 1_{\{z_j\geq z^*_{n'j},n'\neq n\}}}{1+\sum_{n'\neq n}1_{\{z_{nj}=z^*_{n'j}\}}}-\gamma(z)\right]\right\}\\
&=\max_{z\in\mathrm{Im}(W^T)}\left\{\sum_{j=1}^{2}u_j\sum_{\ell=0}^{N-1}\binom{N-1}{\ell}\frac{1}{\ell+1}[\mathbb{P}(z^*_j=z_j)]^{\ell}[\mathbb{P}(z^*_j<z_j)]^{N-1-\ell}-\gamma(z)\right\}\\
&=\max_{z\in\mathrm{Im}(W^T)}\left\{\sum_{j=1}^{2}\frac{u_j}{N}\sum_{\ell=0}^{N-1}\binom{N}{\ell+1}\lim_{\varepsilon\downarrow 0}[G^*_j(z_j)-G^*_j(z_j-\varepsilon)]^{\ell}\lim_{\varepsilon\downarrow 0}[G^*_j(z_j-\varepsilon)]^{N-1-\ell}-\gamma(z)\right\}\\
&=\max_{z\in\mathrm{Im}(W^T)}\left\{\sum_{j=1}^{2}\frac{u_j}{N}\sum_{\ell=1}^{N}\binom{N}{\ell}\lim_{\varepsilon\downarrow 0}[G^*_j(z_j)-G^*_j(z_j-\varepsilon)]^{\ell-1}\lim_{\varepsilon\downarrow 0}[G^*_j(z_j-\varepsilon)]^{N-\ell}-\gamma(z)\right\}\\
&=\max_{z\in\mathrm{Im}(W^T)}\left\{\sum_{j=1}^{2}u_j\lim_{\varepsilon\downarrow 0}\frac{G^{*N}_j(z_j)-G^{*N}_j(z_j-\varepsilon)}{N[G^*_j(z_j)-G^*_j(z_j-\varepsilon)]}-\gamma(z)\right\}.
\end{aligned}
$$

Therefore, by (3), G^* is a symmetric equilibrium if and only if

$$\frac{u_1+u_2}{N}-E_{G^*}[\gamma(\boldsymbol{z}^*)]=\max_{z\in\mathrm{Im}(W^T)}\left\{\sum_{j=1}^{2}u_j\lim_{\varepsilon\downarrow 0}\frac{G^{*N}_j(z_j)-G^{*N}_j(z_j-\varepsilon)}{N[G^*_j(z_j)-G^*_j(z_j-\varepsilon)]}-\gamma(\boldsymbol{z})\right\}.$$

Because

$$
\begin{aligned}
\lim_{\varepsilon\downarrow 0}\frac{G^{*N}_j(z_j)-G^{*N}_j(z_j-\varepsilon)}{N[G^*_j(z_j)-G^*_j(z_j-\varepsilon)]}&=\lim_{\varepsilon\downarrow 0}\frac{1}{N}\sum_{k=0}^{N-1}G^{*k}_j(z_j)G^{*N-1-k}_j(z_j-\varepsilon)\\
&\leq\lim_{\varepsilon\downarrow 0}\frac{1}{N}\sum_{k=0}^{N-1}G^{*k}_j(z_j)G^{*N-1-k}_j(z_j)\\
&=\lim_{\varepsilon\downarrow 0}\frac{1}{N}\sum_{k=0}^{N-1}G^{*N-1}_j(z_j)\\
&=G^{*N-1}_j(z_j)=\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{G^{*N}_j(z_j+\delta)-G^{*N}_j(z_j+\delta-\varepsilon)}{N[G^*_j(z_j+\delta)-G^*_j(z_j+\delta-\varepsilon)]},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\sum_{j=1}^{2}u_j\lim_{\varepsilon\downarrow 0}\frac{G^{*N}_j(z_j)-G^{*N}_j(z_j-\varepsilon)}{N[G^*_j(z_j)-G^*_j(z_j-\varepsilon)]}-\gamma(\boldsymbol{z})\\
&\leq\sum_{j=1}^{2}u_jG^{*N-1}_j(z_j)-\gamma(\boldsymbol{z})\\
&=\lim_{\delta\downarrow 0}\sum_{j=1}^{2}u_j\lim_{\varepsilon\downarrow 0}\frac{G^{*N}_j(z_j+\delta)-G^{*N}_j(z_j+\delta-\varepsilon)}{N[G^*_j(z_j+\delta)-G^*_j(z_j+\delta-\varepsilon)]}-\gamma(\boldsymbol{z}+\delta(1,1)^T)\\
&\leq\max_{z\in\mathrm{Im}(W^T)}\left\{\sum_{j=1}^{2}u_j\lim_{\varepsilon\downarrow 0}\frac{G^{*N}_j(z_j)-G^{*N}_j(z_j-\varepsilon)}{N[G^*_j(z_j)-G^*_j(z_j-\varepsilon)]}-\gamma(z)\right\},
\end{aligned}
$$

where the last inequality holds because $(1,1)^T = \boldsymbol{W}^T\left(\frac{w_1+w_2}{1+\boldsymbol{w}_1^T\boldsymbol{w}_2}\right) \in \mathrm{Im}(\boldsymbol{W}^T)$. Thus,

$$\max_{\boldsymbol{z}\in\mathrm{Im}(\boldsymbol{W}^T)}\left\{\sum_{j=1}^{2} u_j \lim_{\varepsilon\downarrow 0}\frac{G_j^{*N}(z_j)-G_j^{*N}(z_j-\varepsilon)}{N[G_j^*(z_j)-G_j^*(z_j-\varepsilon)]}-\gamma(\boldsymbol{z})\right\} = \max_{\boldsymbol{z}\in\mathrm{Im}(\boldsymbol{W}^T)}\left\{\sum_{j=1}^{2} u_j G_j^{*N-1}(z_j)-\gamma(\boldsymbol{z})\right\},$$

and the support of G^* is a subset of $\arg\max_{\boldsymbol{z}\in\mathrm{Im}(\boldsymbol{W}^T)}\left\{\sum_{j=1}^2 u_j G_j^{*N-1}(z_j)-\gamma(\boldsymbol{z})\right\}$, implying that G^* is a symmetric equilibrium if and only if (A.1) holds.

Moreover, note that for given equilibrium marginal distributions G_1^* and G_2^*, G^* must be an optimal solution to $\min_G\left\{\int\gamma(\boldsymbol{z})\, dG(\boldsymbol{z}) : G \text{ has marginals } (G_1^*, G_2^*)\right\}$, which is a transportation problem with a quadratic cost. By Mitroi and Niculescu [23], z_2^* increases in z_1^* when $\boldsymbol{w}_1^T\boldsymbol{w}_2 > 0$ and decreases in z_1^* when $\boldsymbol{w}_1^T\boldsymbol{w}_2 < 0$. □

Note that Propositions 1–4 identify the equilibria under different contest correlations when there are two contests. This process reduces to finding all solutions to the necessary and sufficient condition (A.1) as established by Lemma A.1. Because (A.1) involves maximizing $\Psi(\boldsymbol{z}) := \sum_{j=1}^2 u_j G_j^{*N-1}(z_j)-\gamma(\boldsymbol{z})$ over the linear subspace $\mathrm{Im}(\boldsymbol{W}^T)$, any equilibrium solution G^* must satisfy the first-order conditions $\frac{\partial\Psi(\boldsymbol{z})}{\partial z_j} = (N-1)u_j G_j^{*N-2}(z_j)G_j^{*'}(z_j) - \frac{\partial\gamma(\boldsymbol{z})}{\partial z_j} = 0$, $j = 1, 2$, which are differential equations for the marginal distributions $G_1^*(z_1)$ and $G_2^*(z_2)$. Thus, we identify the equilibria in the propositions by first finding all of the solutions to the differential equations, which must encompass all potential candidates for the equilibria. We then verify that the equilibria presented in the propositions are indeed those satisfying (A.1).

Proof of Proposition 1. When $\boldsymbol{w}_1^T\boldsymbol{w}_2 > 0$, z_2^* increases in z_1^*, and hence, $G_1^*(z_1^*) = G_2^*(z_2^*)$ by Lemma A.1. Furthermore, it is easy to show that $\mathbf{0}$ is on the support of $\boldsymbol{z}^*$ and $G^*(\mathbf{0}) = 0$.

1. When $\boldsymbol{w}_1^T\boldsymbol{w}_2 = 1$, we have $z_2^* = z_1^*$ and $\gamma(\boldsymbol{z}) = z_1^2$. Then, $\Psi(\boldsymbol{z}) = (u_1+u_2)G_1^{*N-1}(z_1) - z_1^2$, and the first-order condition becomes

$$(N-1)(u_1+u_2)G_1^{*N-2}(z_1^*)G_1^{*'}(z_1^*) = 2z_1^*.$$

This equation has a unique solution:

$$G_1^*(z_1) = \begin{cases} 0, & \text{if } z_1 \le 0,\\ \left(\frac{z_1^2}{u_1+u_2}\right)^{\frac{1}{N-1}}, & \text{if } 0 \le z_1 \le \sqrt{u_1+u_2},\\ 1, & \text{if } z_1 \ge \sqrt{u_1+u_2}, \end{cases}$$

under the boundary condition $G_1^*(0) = 0$. At this solution, we have

$$\Psi(\boldsymbol{z}) = \begin{cases} -z_1^2, & \text{if } z_1 \le 0,\\ 0, & \text{if } 0 \le z_1 \le \sqrt{u_1+u_2},\\ u_1+u_2-z_1^2, & \text{if } z_1 \ge \sqrt{u_1+u_2}. \end{cases}$$

This function is maximized on the entire interval $[0, \sqrt{u_1+u_2}]$. Therefore,

$$\max_{\boldsymbol{z}:z_1=z_2}\{\Psi(\boldsymbol{z})\} + E_{G^*}[\gamma(\boldsymbol{z})] = 0 + \int_{z_1=0}^{\sqrt{u_1+u_2}} z_1^2\, d\left(\frac{z_1^2}{u_1+u_2}\right)^{\frac{1}{N-1}} = \frac{u_1+u_2}{N},$$

and that is, (A.1) holds; hence, G^* is indeed the unique symmetric equilibrium.

2. The first-order conditions

$$u_1\frac{dG_1^{*N-1}(z_1^*)}{dz_1^*} = \frac{2(z_1^*-\boldsymbol{w}_1^T\boldsymbol{w}_2 z_2^*)}{1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2},$$
$$u_2\frac{dG_2^{*N-1}(z_2^*)}{dz_2^*} = \frac{2(z_2^*-\boldsymbol{w}_1^T\boldsymbol{w}_2 z_1^*)}{1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2}$$

can be expressed as

$$\begin{aligned} \frac{dz_2^*}{dz_1^*} &= \frac{u_2(z_1^*-z_2^*\boldsymbol{w}_1^T\boldsymbol{w}_2)}{u_1(z_2^*-z_1^*\boldsymbol{w}_1^T\boldsymbol{w}_2)},\\ u_1\frac{dG_1^{*N-1}(z_1^*)}{dz_1^*} &= \frac{2(z_1^*-\boldsymbol{w}_1^T\boldsymbol{w}_2 z_2^*)}{1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2} \end{aligned} \tag{A.2}$$

because $G_1^*(z_1^*) = G_2^*(z_2^*)$ implies $\frac{dG_1^{*N-1}(z_1^*)}{dz_1^*} = \frac{dG_2^{*N-1}(z_2^*)}{dz_1^*} = \frac{dG_2^{*N-1}(z_2^*)}{dz_2^*}\frac{dz_2^*}{dz_1^*}$. Letting $v(z_1^*) = \frac{z_2^*(z_1^*)}{z_1^*}$ and following Ince [16, section 2.12], we can rewrite the homogeneous ordinary differential equation as

$$v(z_1^*) + v'(z_1^*)z_1^* = \frac{u_2(1 - v(z_1^*)\boldsymbol{w}_1^T\boldsymbol{w}_2)}{u_1(v(z_1^*) - \boldsymbol{w}_1^T\boldsymbol{w}_2)}$$

or equivalently,

$$v'(z_1^*)z_1^* = -\frac{u_1 v^2(z_1^*) - \boldsymbol{w}_1^T\boldsymbol{w}_2(u_1 - u_2)v(z_1^*) - u_2}{u_1(v(z_1^*) - \boldsymbol{w}_1^T\boldsymbol{w}_2)} = -\frac{(v(z_1^*) - v_+)(v(z_1^*) - v_-)}{v(z_1^*) - \boldsymbol{w}_1^T\boldsymbol{w}_2},$$

where $v_- < 0 < v_+$ are the two distinct solutions to the quadratic equation $u_1 v^2 - \boldsymbol{w}_1^T\boldsymbol{w}_2(u_1 - u_2)v - u_2 = 0$. The solutions to the equation can be $v(z_1^*) = v_+$, $v(z_1^*) = v_-$, or $z_1^* = C(v(z_1^*) - v_-)^{-\frac{v_- - \boldsymbol{w}_1^T\boldsymbol{w}_2}{v_- - v_+}}|v_+ - v(z_1^*)|^{-\frac{v_+ - \boldsymbol{w}_1^T\boldsymbol{w}_2}{v_+ - v_-}}$ for some constant C, among which only the first one includes $\mathbf{0}$ in the support as well as satisfying $G^*(\mathbf{0}) = 0$. Consequently, we have a unique candidate: $z_2^* = v_+ z_1^*$,

$$G_1^*(z_1) = \begin{cases} 0, & \text{if } z_1 \le 0, \\ \left(\dfrac{(1 - \boldsymbol{w}_1^T\boldsymbol{w}_2 v_+)z_1^2}{u_1(1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)}\right)^{\frac{1}{N-1}}, & \text{if } 0 \le z_1 \le \sqrt{\dfrac{u_1(1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)}{1 - \boldsymbol{w}_1^T\boldsymbol{w}_2 v_+}}, \\ 1, & \text{if } z_1 \ge \sqrt{\dfrac{u_1(1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)}{1 - \boldsymbol{w}_1^T\boldsymbol{w}_2 v_+}}, \end{cases}$$

$$G_2^*(z_2) = \begin{cases} 0, & \text{if } z_2 \le 0, \\ \left(\dfrac{(1 - \boldsymbol{w}_1^T\boldsymbol{w}_2 v_+)z_2^2}{u_1 v_+^2(1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)}\right)^{\frac{1}{N-1}}, & \text{if } 0 \le z_2 \le v_+\sqrt{\dfrac{u_1(1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)}{1 - \boldsymbol{w}_1^T\boldsymbol{w}_2 v_+}}, \\ 1, & \text{if } z_2 \ge v\sqrt{\dfrac{u_1(1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)}{1 - \boldsymbol{w}_1^T\boldsymbol{w}_2 v_+}}, \end{cases}$$

where $v_+ = \frac{(u_1 - u_2)\boldsymbol{w}_1^T\boldsymbol{w}_2 + \sqrt{[(u_1 - u_2)\boldsymbol{w}_1^T\boldsymbol{w}_2]^2 + 4u_1u_2}}{2u_1}$. Letting

$$\tilde{z}_1 = \max\left\{0, \min\left\{\sqrt{\frac{u_1(1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)}{1 - \boldsymbol{w}_1^T\boldsymbol{w}_2 v_+}}, z_1\right\}\right\},$$

$$\tilde{z}_2 = \max\left\{0, \min\left\{v_+\sqrt{\frac{u_1(1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)}{1 - \boldsymbol{w}_1^T\boldsymbol{w}_2 v_+}}, z_2\right\}\right\},$$

we have at this solution

$$\begin{aligned} \Psi(\boldsymbol{z}) &= \sum_{j=1}^{2} u_j G^{*N-1}(z_j) - \gamma(\boldsymbol{z}), \\ &= \sum_{j=1}^{2} u_j G^{*N-1}(z_j) - \gamma(\tilde{\boldsymbol{z}}) - [\gamma(\boldsymbol{z}) - \gamma(\tilde{\boldsymbol{z}})], \\ &= \frac{u_1(1 - \boldsymbol{w}_1^T\boldsymbol{w}_2 v_+)v_+^2\tilde{z}_1^2 + u_2(1 - \boldsymbol{w}_1^T\mathbf{w}_2 v_+)\tilde{z}_2^2 - u_1 v_+^2(\tilde{z}_1^2 + \tilde{z}_2^2 - 2\boldsymbol{w}_1^T\boldsymbol{w}_2\tilde{z}_1\tilde{z}_2)}{u_1 v_+^2(1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)} - [\gamma(\boldsymbol{z}) - \gamma(\tilde{\boldsymbol{z}})] \\ &= -\frac{\boldsymbol{w}_1^T\boldsymbol{w}_2(v_+\tilde{z}_1 - \tilde{z}_2)^2}{v_+(1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)} - [\gamma(\boldsymbol{z}) - \gamma(\tilde{\boldsymbol{z}})], \end{aligned}$$

which is maximized at $z_1^* = v_+ z_2^*$ for $z_1^* \in \left[0, \sqrt{\frac{u_1(1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2)}{1 - \boldsymbol{w}_1^T\boldsymbol{w}_2 v}}\right]$ as $\boldsymbol{w}_1^T\boldsymbol{w}_2 \le v_+ \le \frac{1}{\boldsymbol{w}_1^T\boldsymbol{w}_2}$. Therefore, (A.1) holds, proving that G^* is indeed the unique symmetric equilibrium by Lemma A.1. □

Proof of Proposition 2. In this case, it is straightforward to show that $\mathbf{0}$ is within the support of $\boldsymbol{z}^*$ and that $G^*(\mathbf{0}) = 0$. By Lemma A.1, when $\boldsymbol{w}_1^T\boldsymbol{w}_2 = 0$, the first-order condition yields two independent differential equations:

$$(N-1)u_j G_j^{*N-2}(z_j^*)G_j^{*'}(z_j^*) = z_j^*, \quad j = 1, 2,$$

each with a unique solution given by

$$G_j^*(z_j) = \begin{cases} 0, & \text{if } z_j \le 0, \\ \left(\frac{z_j^2}{u_j}\right)^{\frac{1}{N-1}}, & \text{if } 0 \le z_j \le \sqrt{u_j}, \\ 1, & \text{if } z_j \ge \sqrt{u_j}, \end{cases}$$

under the boundary condition $G_j^*(0) = 0$. Thus, an equilibrium, if it exists, must have the above marginals (G_1^*, G_2^*). Furthermore, at this solution, we have

$$\Psi(z) = \sum_{j=1}^{2} \max\{0, \min\{u_j, z_j^2\}\} - z_j^2,$$

which is maximized at $z_j^* \in [0, \sqrt{u_j}]$. Hence, (A.1) holds, confirming that G^* is indeed an equilibrium by Lemma A.1. □

Proof of Proposition 3. When $w_1^T w_2 = -1$, it holds that $z_1^* = -z_2^*$.

1. When $N = 2$ and $u_1 = u_2$, for any distribution G^*, $z^* = \mathbf{0}$ is a maximizer of (3) (i.e., $z^* = \mathbf{0}$ is a dominant strategy) and hence, the unique symmetric equilibrium.

2. Otherwise, by (3), $G_1^*(z_1^*) = 1 - G_2^*(-z_1^*)$ is an equilibrium if and only if

$$\max_{z_1}\left\{\lim_{\varepsilon\downarrow 0}\frac{u_1[G_1^{*N}(z_1) - G_1^{*N}(z_1-\varepsilon)] + u_2[(1-G_1^*(z_1-\varepsilon))^N - (1-G_1^*(z_1))^N]}{N[G_1^*(z_1) - G_1^*(z_1-\varepsilon)]} - z_1^2\right\} = \frac{u_1+u_2}{N} - E_{G_1^*}[z_1^{*2}].$$

Because

$$\lim_{\varepsilon\downarrow 0}\frac{u_1[G_1^{*N}(z_1) - G_1^{*N}(z_1-\varepsilon)] + u_2[(1-G_1^*(z_1-\varepsilon))^N - (1-G_1^*(z_1))^N]}{N[G_1^*(z_1) - G_1^*(z_1-\varepsilon)]}$$
$$\le \frac{1}{2}\left\{u_1 G_1^{*N-1}(z_1) + u_2[1-G_1^*(z_1)]^{N-1} + \lim_{\varepsilon\downarrow 0}[u_1 G_1^{*N-1}(z_1-\varepsilon) + u_2(1-G_1^*(z_1-\varepsilon))^{N-1}]\right\}$$

when $N \ne 2$ or $u_1 \ne u_2$, G_1^* is an equilibrium if and only if

$$\max_{z_1}\{u_1 G_1^{*N-1}(z_1) + u_2[1-G_1^*(z_1)]^{N-1} - z_1^2\} + E_{G_1^*}[z_1^{*2}] = \frac{u_1+u_2}{N}. \tag{A.3}$$

Furthermore, it is straightforward to show that the support of G_1^* is a subset of $\arg\max_{z_1}\{u_1 G_1^{*N-1}(z_1) + u_2(1-G_1^*(z_1))^{N-1} - z_1^2\}$ and must include zero. Thus, we must have

$$u_1 G_1^{*N-1}(z_1^*) + u_2(1-G_1^*(z_1^*))^{N-1} - z_1^{*2} = u_1 G_1^{*N-1}(0) + u_2(1-G_1^*(0))^{N-1} \tag{A.4}$$

for all z_1^* in the support of G^*. It suffices to find $G^*(0)$ such that G_1^* increases from zero to one on its support and that (A.3) holds.

a. If $N = 2$ and $u_1 > u_2$, (A.4) becomes $G_1^*(z_1) = G^*(0) + \frac{z_1^2}{u_1-u_2}$. Because $u_1 > u_2$ and $G_1^*(z_1)$ must increase from zero to one on its support, we have

$$G_1^*(z_1) = \begin{cases} 0, & \text{if } z_1 \le 0, \\ \dfrac{z_1^2}{u_1-u_2}, & \text{if } 0 \le z_1 \le \sqrt{u_1-u_2}, \\ 1, & \text{if } z_1 \ge \sqrt{u_1-u_2}. \end{cases}$$

At this solution,

$$u_1 G_1^*(z_1) + u_2(1-G_1^*(z_1)) - z_1^2 = \begin{cases} u_2 - z_1^2, & \text{if } z_1 \le 0, \\ u_2, & \text{if } 0 \le z_1 \le \sqrt{u_1-u_2}, \\ u_1 - z_1^2, & \text{if } z_1 \ge \sqrt{u_1-u_2}. \end{cases}$$

Hence,

$$\max_{z_1}\{u_1 G_1^*(z_1) + u_2(1-G_1^*(z_1)) - z_1^2\} + E_{G_1^*}[z_1^{*2}] = u_2 + \int_0^{\sqrt{u_1-u_2}} z_1^2 d\frac{z_1^2}{u_1-u_2} = \frac{u_1+u_2}{2},$$

and that is, (A.3) holds; additionally, G_1^* is indeed the unique symmetric equilibrium.

b. If $N > 2$, let $\hat{u} = u_1 G_1^{*N-1}(0) + u_2(1-G_1^*(0))^{N-1} = u_1 G_1^{*N-1}(z_1^*) + u_2(1-G_1^*(z_1^*))^{N-1} - z_1^{*2}$ for z_1^* in the support of G_1^*. Then, we can express z_1^{*2} as a convex function of G_1^* as $z_1^{*2} = u_1 G_1^{*N-1} + u_2(1-G_1^*)^{N-1} - \hat{u}$, which is minimized at $G_1^* = \frac{\sqrt[N-2]{u_2}}{\sqrt[N-2]{u_1} + \sqrt[N-2]{u_2}}$ with a minimum value of $z_1^{*2} = 0$. This implies that $G_1^*(0) = \frac{\sqrt[N-2]{u_2}}{\sqrt[N-2]{u_1} + \sqrt[N-2]{u_2}}$ and $\hat{u} = u_1\left(\frac{\sqrt[N-2]{u_2}}{\sqrt[N-2]{u_1} + \sqrt[N-2]{u_2}}\right)^{N-1} + u_2\left(\frac{\sqrt[N-2]{u_1}}{\sqrt[N-2]{u_1} + \sqrt[N-2]{u_2}}\right)^{N-1}$. Because $u_1 > u_2$ and $G_1^*(z_1)$ must increase from zero to one on its support,

$$u_1 G_1^{*N-1}(z_1) + u_2(1 - G_1^*(z_1))^{N-1} - z_1^2 = \begin{cases} u_2 - z_1^2, & \text{if } z_1 \leq -\sqrt{\hat{u} - u_2}, \\ \hat{u}, & \text{if } -\sqrt{\hat{u} - u_2} \leq z_1 \leq \sqrt{u_1 - \hat{u}}, \\ u_1 - z_1^2, & \text{if } z_1 \geq \sqrt{u_1 - \hat{u}}. \end{cases}$$

Therefore, (A.3) holds at G_1^* as

$$\begin{aligned} &\max_{z_1} \{u_1 G_1^*(z_1) + u_2(1 - G_1^*(z_1)) - z_1^2\} + E_{G_1^*}[z_1^{*2}] \\ &= \hat{u} + \int_{-\sqrt{\hat{u}-u_2}}^{\sqrt{u_1-\hat{u}}} z_1^2 dG_1^*(z_1) \\ &= \hat{u} + \int_{-\sqrt{\hat{u}-u_2}}^{\sqrt{u_1-\hat{u}}} [u_1 G_1^{*N-1}(z_1^*) + u_2(1 - G_1^*(z_1^*))^{N-1} - \hat{u}] dG_1^*(z_1) \\ &= \frac{u_1 + u_2}{N}. \quad \square \end{aligned}$$

Proof of Proposition 4. By Lemma A.1, z_2^* is decreasing in z_1^*, and hence, $G_1^*(z_1^*) + G_2^*(z_2^*) = 1$. Letting $t = G_1^*(z_1^*) \in [0,1]$, we have $(z_1^*(t), z_2^*(t)) = (G_1^{*-1}(t), G_2^{*-1}(1-t))$, and the first-order conditions

$$(N-1)u_1 t^{N-2} = \frac{2}{1 - (\boldsymbol{w}_1^T \boldsymbol{w}_2)^2}(z_1^* - z_2^* \boldsymbol{w}_1^T \boldsymbol{w}_2) z_1^{*'}(t),$$

$$-(N-1)u_2(1-t)^{N-2} = \frac{2}{1 - (\boldsymbol{w}_1^T \boldsymbol{w}_2)^2}(z_2^* - z_1^* \boldsymbol{w}_1^T \boldsymbol{w}_2) z_2^{*'}(t).$$

Furthermore, because $G_1^*(z_1^*(0)) = G_2^*(z_2^*(1)) = 0$, we have

$$u_1 G_1^{*N-1}(\boldsymbol{w}_1^T \boldsymbol{w}_2 z_2^*(0)) + u_2 G_2^{*N-1}(z_2^*(0)) - \gamma((\boldsymbol{w}_1^T \boldsymbol{w}_2 z_2^*(0), z_2^*(0))) \geq u_1 G_1^{*N-1}(z_1^*(0)) + u_2 G_2^{*N-1}(z_2^*(0)) - \gamma(\boldsymbol{z}^*(0)),$$
$$u_1 G_1^{*N-1}(z_1^*(1)) + u_2 G_2^{*N-1}(\boldsymbol{w}_1^T \boldsymbol{w}_2 z_1^*(1)) - \gamma((z_1^*(1), \boldsymbol{w}_1^T \boldsymbol{w}_2 z_2^*(1))) \geq u_1 G_1^{*N-1}(z_1^*(1)) + u_2 G_2^{*N-1}(z_2^*(1)) - \gamma(\boldsymbol{z}^*(1)),$$

which lead to the boundary equations $\frac{z_2^*(1)}{z_1^*(1)} = \frac{z_1^*(0)}{z_2^*(0)} = \boldsymbol{w}_1^T \boldsymbol{w}_2$.

Letting $r(t) = \sqrt{\boldsymbol{z}^*(t)(\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{z}^*(t)} = \sqrt{\gamma(\boldsymbol{z}^*(t))}$, we have $r^2(t) = \boldsymbol{z}^*(t)^T(\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{z}^*(t) = (\boldsymbol{Q}\boldsymbol{z}^*(t))^T(\boldsymbol{Q}\boldsymbol{z}^*(t))$, where $\boldsymbol{Q} = \frac{1}{\sqrt{1-(\boldsymbol{w}_1^T\boldsymbol{w}_2)^2}} \begin{pmatrix} \sqrt{\frac{1-\boldsymbol{w}_1^T\boldsymbol{w}_2}{2}} & \sqrt{\frac{1-\boldsymbol{w}_1^T\boldsymbol{w}_2}{2}} \\ \sqrt{\frac{1+\boldsymbol{w}_1^T\boldsymbol{w}_2}{2}} & -\sqrt{\frac{1+\boldsymbol{w}_1^T\boldsymbol{w}_2}{2}} \end{pmatrix}$. Therefore, we have $\boldsymbol{Q}\boldsymbol{z}^*(t) = r(t)(\cos(\theta(t)), \sin(\theta(t)))^T$, which leads to $\theta(t) = \arcsin\left(\frac{z_1^*(t) - z_2^*(t)}{2r(t)\sin(\theta_0)}\right)$ and $\theta_0 = \arcsin\left(\sqrt{\frac{1-\boldsymbol{w}_1^T\boldsymbol{w}_2}{2}}\right)$. Thus, $z_1^*(t) = r(t)\cos(\theta(t) - \theta_0)$ and $z_2^*(t) = r(t)\cos(\theta(t) + \theta_0)$. The first-order condition can be rewritten as

$$\begin{aligned} r^2(t) &= u_1 t^{N-1} + u_2(1-t)^{N-1} - \frac{u_1 + u_2}{N} + \lambda \\ \frac{\theta'(t)}{N-1} &= \frac{u_1 t^{N-2}\cot(\theta + \theta_0) + u_2(1-t)^{N-2}\cot(\theta_0 - \theta)}{2\left(u_1 t^{N-1} + u_2(1-t)^{N-1} - \frac{u_1+u_2}{N} + \lambda\right)}, \end{aligned} \tag{A.5}$$

and the boundary conditions become $\theta_0 = \theta(1) = -\theta(0)$.

Lemma A.2 below establishes that Equation (A.5) has a unique $(\theta(t), \lambda)$ under the boundary conditions, and hence, an equilibrium, if it exists, must be unique.

It remains to verify that the above G^* is indeed an equilibrium. Because the Hessian of $\sum_{j=1}^2 u_j G_j^{*N-1}(z_j) - \gamma(\boldsymbol{z})$ at any point $\boldsymbol{z}^*$ satisfying the first-order condition

$$\frac{-2\boldsymbol{w}_1^T\boldsymbol{w}_2}{1 - (\boldsymbol{w}_1^T\boldsymbol{w}_2)^2} \begin{pmatrix} \frac{dz_2^*}{dz_1^*} & -1 \\ -1 & \frac{dz_1^*}{dz_2^*} \end{pmatrix}$$

is negative semidefinite when $-1 < \boldsymbol{w}_1^T\boldsymbol{w}_2 < 0$, (A.1) holds at G^*, confirming that G^* is indeed an equilibrium. $\square$

Lemma A.2. *Let* $t_0 = \left(\frac{\sqrt[N-2]{u_2}}{\sqrt[N-2]{u_1}+\sqrt[N-2]{u_2}}\right)$. *There is a unique* $(\theta(t), \lambda)$ *such that* $\lambda > \frac{u_1+u_2}{n} - u_1 t_0^{N-1} - u_2(1-t)^{N-1}$, $\lim_{t\downarrow 0}\theta(t) = -\lim_{t\uparrow 1}\theta(t) = -\theta_0$, *and* (A.5) *holds for* $t \in (0,1)$.

Proof of Lemma A.2. First, we show that for each $\lambda > 0$, a unique solution exists, denoted as $\theta_\infty(t)$, to (A.5) such that $\lim_{t\downarrow 0}\theta_\infty(t) = -\theta_0$. As for any $(t,\theta) \in [0,1] \times [-\theta_0, \theta_0] \setminus \{(0,-\theta_0),(1,\theta_0)\}$, the right-hand side of (A.5) is locally Lipschitz in θ (or its reciprocal is locally Lipschitz in t), a unique solution $\theta(\cdot)$ to (A.5) exists such that $\theta(t) = \theta$ by Picard's theorem. Let $\theta_k(\cdot)$ be the unique solution to (A.5) passing through $(t,\theta) = \left(0, \left(-1+\frac{1}{k}\right)\theta_0\right)$ for $k = 1,2,\ldots$. Then, $\theta_k(t)$ is decreasing in k and thus, point-wise converges to a function, which we will show is the unique solution $\theta_\infty(t)$ to (A.5) such that $\lim_{t\downarrow 0}\theta_\infty(t) = -\theta_0$.

For $t \in [\varepsilon, 2\varepsilon]$, where $\varepsilon > 0$ are small enough, $\theta_k(t)$ is bounded from below by the solution to (A.5) passing through $\left(\frac{\varepsilon}{2}, -\theta_0\right)$. Thus, $\theta_k'(t)$ is uniformly convergent as $k \to \infty$. As $\limsup_{t\downarrow 0}\theta_\infty(t) \le \lim_{t\downarrow 0}\theta_k(t) = \left(-1+\frac{1}{k}\right)\theta_0$ for all k, $\lim_{t\downarrow 0}\theta_\infty(t) = -\theta_0$, and $\theta_\infty(t)$ is a solution.

Any solution to (A.5) starts with $(t,\theta) = (0,-\theta_0)$ and will never cross other solutions even if they exist. As the right-hand side of (A.5) is decreasing in θ when t and θ are small enough, θ_∞ is the unique solution.

We then show that there exists a unique $\lambda > \frac{u_1+u_2}{n} - u_1 t_0^{n-1} - u_2(1-t_0)^{n-1}$, at which $\theta_\infty(1) = \theta_0$. By (A.5), $\frac{d\theta}{dt} \le \frac{(n-1)(u_1+u_2)\cot(\theta+\theta_0)}{\lambda}$ for $\theta < 0$. As $\frac{d\theta}{dt} = \frac{(n-1)(u_1+u_2)\cot(\theta+\theta_0))}{\lambda}$, where $\theta(0) = -\theta_0$ has a solution $\cos(\theta(t)+\theta_0) = e^{-\frac{(n-1)(u_1+u_2)}{\lambda}t}$, $\theta(1) < 0$, and $\theta_\infty(t)$ intersects with the line $t = 1$ for λ^* that is large enough.

Let $\lambda(\varepsilon) = \frac{u_1+u_2}{N} - u_1 t_0^{N-1} - u_2(1-t_0)^{N-1} + \varepsilon$. For $t \in [t_0-\delta, t_0+\delta]$, where $\delta > 0$ is small enough,

$$\theta(t_0+\delta) \ge -\theta_0 + \min_{t\in[t_0-\delta,t_0+\delta]}\{\theta_\infty'(t)2\delta\}$$

by the mean-value theorem, and

$$\lim_{\varepsilon\downarrow 0}\min_{t\in[t_0-\delta,t_0+\delta]}\left\{\theta_\infty'(t)2\delta\right\} \ge \lim_{\varepsilon\downarrow 0}\left\{\frac{n-1}{2}\frac{\min\limits_{t\in[t_0-\delta,t_0+\delta],\,\theta\in[-\theta_0,\theta_0]}\{u_1 t^{N-2}\cot(\theta+\theta_0) + u_2(1-t)^{N-2}\cot(\theta_0-\theta)\}}{\max\limits_{t\in[t_0-\delta,t_0+\delta]}\left\{u_1 t_0^{N-1} + u_2(1-t)^{N-1} - \dfrac{u_1+u_2}{N} + \lambda(\varepsilon)\right\}}2\delta\right\}$$
$$= \frac{\cot(\theta_0)[u_1 t_0^{N-2} + u_2(1-t_0)^{N-2}] + O(\delta)}{O(\delta^2)}\delta \to \infty, \text{ as } \delta \to 0.$$

Thus, there exist positive ε and δ such that $\theta_\infty(t) = \theta_0$ at $\lambda = \lambda(\varepsilon)$ for some $t \le t_0 + \delta$. Thus, for λ that is small enough, $\theta_\infty(t)$ intersects with the line $\theta = \theta_0$.

Because $\theta_\infty(t)$ and hence, its unique intersection with $t = 1$ or $\theta = \theta_0$ are continuous in λ, a λ exists such that $\theta_\infty(1) = \theta_0$. Furthermore, it can be shown by contradiction that $\theta_\infty(t)$ passes through $(1,\theta_0)$ only if $\theta_\infty'(t) \ge 0$ for all t by (A.5). Thus, $\theta_\infty'(t)$ is decreasing in λ, and there exists a unique λ such that $\theta_\infty(1) = \theta_0$. □

Proof of Proposition 5. Let

$$\hat{z}(t) = \begin{cases} \hat{r}(t)(w_1^T w_2, 1), & \text{if } t \in [0, t_0], \\ \hat{r}(t)(1, w_1^T w_2), & \text{if } t \in [t_0, 1], \end{cases}$$

and $\hat{G}_1(\hat{z}_1(t)) = 1 - \hat{G}_2(\hat{z}_2(t)) = t$, where $\hat{r}(t) = \sqrt{u_1 t^{N-1} + u_2(1-t)^{N-1} - u_1 t_0^{N-1} - u_2(1-t_0)^{N-1}}$. Recall that $(\lambda, \theta(t))$ is the unique solution to (A.5) with the boundary conditions $\theta(0) = -\theta(1) = -\theta_0$, and note that $r(t) = \sqrt{u_1 t^{N-1} + u_2(1-t)^{N-1} - \frac{u_1+u_2}{N} + \lambda}$ and $\hat{r}(t) = \sqrt{u_1 t^{N-1} + u_2(1-t)^{N-1} - u_1 t_0^{N-1} - u_2(1-t_0)^{N-1}}$. Let $\hat{\theta}(t) = -\theta_0$ when $t \in [0, t_0]$ and $\hat{\theta}(t) = \theta_0$ when $t \in (t_0, 1]$. As $\theta'(t) \ge 0$, we have

$$t \ge \frac{\sqrt[N-2]{u_2\cot(\theta(t)-\theta_0)}}{\sqrt[N-2]{u_1\cot(\theta(t)+\theta_0)} + \sqrt[N-2]{u_2\cot(\theta(t)-\theta_0)}}, \quad \text{if } t \in \left[0, \frac{\sqrt[N-2]{u_2\cot\left(-\frac{\pi}{4}-\theta_0\right)}}{\sqrt[N-2]{u_1\cot\left(\theta_0-\frac{\pi}{4}\right)} + \sqrt[N-2]{u_2\cot\left(-\frac{\pi}{4}-\theta_0\right)}}\right],$$

$$t \le \frac{\sqrt[N-2]{u_2\cot(\theta(t)-\theta_0)}}{\sqrt[N-2]{u_1\cot(\theta(t)+\theta_0)} + \sqrt[N-2]{u_2\cot(\theta(t)-\theta_0)}}, \quad \text{if } t \in \left[\frac{\sqrt[N-2]{u_2\cot\left(\frac{\pi}{4}-\theta_0\right)}}{\sqrt[N-2]{u_1\cot\left(\theta_0+\frac{\pi}{4}\right)} + \sqrt[N-2]{u_2\cot\left(\frac{\pi}{4}-\theta_0\right)}}, 1\right].$$

Thus, $\theta(t) + \theta_0 = O(3^{-N})$ for $t \in \left[0, \frac{1}{4}\right]$, and $\theta_0 - \theta(t) = O(3^{-N})$ for $t \in \left[\frac{3}{4}, 1\right]$. Furthermore, according to the mean-value theorem,

$$\frac{\frac{\pi}{8} - \left(-\frac{\pi}{8}\right)}{\frac{\sqrt[N-2]{u_2\cot\left(\frac{\pi}{4}-\theta_0\right)}}{\sqrt[N-2]{u_1\cot\left(\theta_0+\frac{\pi}{4}\right)} + \sqrt[N-2]{u_2\cot\left(\frac{\pi}{4}-\theta_0\right)}} - \frac{\sqrt[N-2]{u_2\cot\left(-\frac{\pi}{4}-\theta_0\right)}}{\sqrt[N-2]{u_1\cot\left(\theta_0-\frac{\pi}{4}\right)} + \sqrt[N-2]{u_2\cot\left(-\frac{\pi}{4}-\theta_0\right)}}} \le \frac{(N-1)(u_1 t_0^{N-2} + u_2(1-t_0)^{N-2})O(1)}{\lambda - \frac{u_1+u_2}{N} + u_1 t_0^{N-1} + u_2(1-t_0)^{N-1}},$$

and thus, $\lambda - \frac{u_1+u_2}{N} + u_1 t_0^{N-1} + u_2(1-t_0)^{N-1} = O(2^{-N})$. Therefore,

$$\sup_{t\in[0,\frac{1}{4}]} \|(r(t)\cos\theta(t), r(t)\sin\theta(t)) - (\hat{r}(t)\cos\hat{\theta}(t), \hat{r}(t)\sin\hat{\theta}(t))\|_2$$
$$\leq O(\hat{r}(0)(\theta(t)+\theta_0)) + \sup_{t\in[0,\frac{1}{4}]} |r(t)-\hat{r}(t)| = O\left(\frac{\sqrt{N}}{3^N}\right),$$
$$\sup_{t\in[\frac{1}{4},\frac{3}{4}]} \|(r(t)\cos\theta(t), r(t)\sin\theta(t)) - (\hat{r}(t)\cos\hat{\theta}(t), \hat{r}(t)\sin\hat{\theta}(t))\|_2$$
$$\leq \sup_{t\in[\frac{1}{4},\frac{3}{4}]} \{r(t)+\hat{r}(t)\} = O\left(\sqrt{N}\left(\frac{\sqrt{3}}{2}\right)^N\right),$$
$$\sup_{t\in[\frac{3}{4},1]} \|(r(t)\cos\theta(t), r(t)\sin\theta(t)) - (\hat{r}(t)\cos\hat{\theta}(t), \hat{r}(t)\sin\hat{\theta}(t))\|_2$$
$$\leq O(\hat{r}(1)(\theta_0-\theta(t))) + \sup_{t\in[\frac{3}{4},1]} |r(t)-\hat{r}(t)| = O\left(\frac{\sqrt{N}}{3^N}\right),$$

and hence,

$$\sup_{t\in[0,1]} \|(r(t)\cos\theta(t), r(t)\sin\theta(t)) - (\hat{r}(t)\cos\hat{\theta}(t), \hat{r}(t)\sin\hat{\theta}(t))\| = O\left(\max\left\{\frac{\sqrt{N}}{3^N}, \sqrt{N}\left(\frac{\sqrt{3}}{2}\right)^N\right\}\right) = O\left(\sqrt{N}\left(\frac{3}{4}\right)^{\frac{N}{2}}\right).$$

Letting $\hat{\boldsymbol{Z}} = \hat{z}(U)$ and $\boldsymbol{Z}^* = (r(U)\cos\theta(U), r(U)\sin\theta(U))$, where U is a uniform random variable on [0, 1], we have $\hat{\boldsymbol{Z}} \sim \hat{G}$, $\boldsymbol{Z}^* \sim G^*$, and $\|\boldsymbol{Z}^* - \hat{\boldsymbol{Z}}\| = O\left(\sqrt{N}\left(\frac{3}{4}\right)^{\frac{N}{2}}\right)$. □

Proofs of Propositions 6 and 11. Because Proposition 6 is a special case of Proposition 11, we will prove the latter. In this case, the objective in (3) becomes

$$\sum_{j=1}^{J} u_j \int H_j^{N-1}\left(\frac{z_j + \beta_j\epsilon - z_j^*}{\beta_j}\right) h(\epsilon)d\epsilon - \gamma(\boldsymbol{z}).$$

When $(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_N)$ are linearly independent, the first-order condition yields that

$$(N-1)\left(\frac{u_1\eta_1}{\beta_1}, \ldots, \frac{u_N\eta_N}{\beta_N}\right)^T = 2(\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{z}^*,$$

where $\eta_j = \int_\epsilon H_j^{N-1}(\epsilon)h_j^2(\epsilon)d\epsilon$. Hence, the unique pure-strategy equilibrium must be $\boldsymbol{z}^* = \frac{N-1}{2}(\boldsymbol{W}^T\boldsymbol{W})\left(\frac{u_1\eta_1}{\beta_1}, \ldots, \frac{u_N\eta_N}{\beta_N}\right)^T$ or $\boldsymbol{x}^* = \frac{N-1}{2}\boldsymbol{W}\left(\frac{u_1\eta_1}{\beta_1}, \ldots, \frac{u_N\eta_N}{\beta_N}\right)^T = \frac{N-1}{2}\sum_{j=1}^{J}\frac{u_j\eta_j}{\beta_j}\boldsymbol{w}_j$ if it exists. To show that the above $\boldsymbol{z}^*$ is indeed an equilibrium, note that the Hessian converges to $-(\boldsymbol{W}^T\boldsymbol{W})^{-1}$ as $\min\{\beta_1, \ldots, \beta_J\} \to \infty$ and hence, is negative definite when all of the β_j's are large enough. □

Proof of Proposition 7. We will demonstrate the existence of the equilibria by verifying that the solutions in the proposition meet the equilibrium conditions and then, establish their uniqueness.

1a. Existence. At $z^* = \frac{u}{2\beta}$,

$$\hat{G}^*(\hat{z}^*) = \min\left\{1, \max\left\{0, \frac{1}{2} + \frac{1}{\beta}\left(\hat{z}^* - \frac{u}{2\beta}\right)\left(1 - \frac{1}{2\beta}\cdot\left|\hat{z}^* - \frac{u}{2\beta}\right|\right)\right\}\right\}.$$

Because $\frac{u}{2\beta}$ is also the unique solution to the first-order optimality condition (i.e., $\max\left\{0, \frac{u}{\beta} - \frac{u}{2\beta^2}\left|\hat{z}^* - \frac{u}{2\beta}\right|\right\} - 2\hat{z}^* = 0$) of a contest's objective $u\hat{G}^*(\hat{z}^*) - \hat{z}^{*2}$, we have that $z^* = \frac{u}{2\beta}$ is indeed an equilibrium.

1b. Uniqueness. Suppose that there exists another equilibrium. Then, any convex combination of the two strategies is also a solution to the convex Programs (5) and (6) and hence, a mixed-strategy equilibrium, referred to as G^*, whose support includes $z^* = \frac{u}{2\beta}$. Because any point in the support of G^*, $\hat{z}^*$, must be a maximizer of a contest's objective function $u\hat{G}^*(\hat{z}) - \hat{z}^2$, it satisfies the first-order optimality condition

$$u\hat{G}^{*'}(\hat{z}^*) - 2\hat{z}^* = u\int_{\hat{z}^*-\beta}^{\hat{z}^*+\beta}\left(\frac{1}{\beta} - \frac{|\hat{z}^*-z|}{\beta^2}\right)dG^*(z) - 2\hat{z}^* = 0$$

or

$$\hat{z}^* = \frac{u}{2}\int_{\hat{z}^*-\beta}^{\hat{z}^*+\beta}\left(\frac{1}{\beta} - \frac{|\hat{z}^*-z|}{\beta^2}\right)dG^*(z) = \frac{u}{2}E\left[\left(\frac{1}{\beta} - \frac{|\hat{z}^*-\hat{Z}^*|}{\beta^2}\right)1_{\{\hat{Z}^*\in[\hat{z}^*-\beta,\hat{z}^*+\beta]\}}\right] \leq \frac{u}{2\beta}.$$

The equality holds if and only if $\mathbb{P}(\hat{z}^* = \frac{u}{2\beta}) = 1$. Thus, G^* cannot be a mixed-strategy equilibrium, and the pure strategy with $z^* = \frac{u}{2\beta}$ is the unique equilibrium.

2a. Existence. When $\mathbb{P}(z^* = z) = \frac{2\beta}{u} z$ for any $z \in \left\{\frac{u}{2(k+1)\beta} + (\ell - \frac{k}{2})\beta : \ell = 0, 1, \ldots, k\right\}$,

$$\hat{G}^*(\hat{z}^*) = \begin{cases} \left[\dfrac{1}{2(k+1)\beta^2} - \dfrac{k}{2u}\right] \max\left\{0, z^* - \dfrac{u}{2(k+1)\beta} + \left(1 + \dfrac{k}{2}\right)\beta\right\}^2, & \text{if } \hat{z}^* < \dfrac{u}{2(k+1)\beta} - \dfrac{k\beta}{2}, \\ \dfrac{1}{2(k+1)} - \dfrac{k\beta^2}{2u} + \dfrac{z^{*2}}{u} - \dfrac{1}{u}\left[\dfrac{u}{2(k+1)\beta} - \dfrac{k\beta}{2}\right]^2, & \text{if } \dfrac{u}{2(k+1)\beta} - \dfrac{k\beta}{2} \le \hat{z}^* < \dfrac{u}{2(k+1)\beta} + \dfrac{k\beta}{2}, \\ 1 - \left[\dfrac{1}{2(k+1)\beta^2} + \dfrac{k}{2u}\right] \max\left\{0, \dfrac{u}{2(k+1)\beta} + \dfrac{k\beta}{2} + \beta - z^*\right\}^2, & \text{if } \hat{z}^* \ge \dfrac{u}{2(k+1)\beta} + \dfrac{k\beta}{2}. \end{cases}$$

Because any point in the interval $\left[\frac{u}{2(k+1)\beta} - \frac{k\beta}{2}, \frac{u}{2(k+1)\beta} + \frac{k\beta}{2}\right] \supset \left\{\frac{u}{2(k+1)\beta} + (\ell - \frac{k}{2})\beta : \ell = 0, 1, \ldots, k\right\}$ is a maximizer of a competitor's objective $u\hat{G}^*(\hat{z}^*) - \hat{z}^{*2}$, G^* is indeed an equilibrium.

2b. Uniqueness. Following the same argument as that in part 1, given any equilibrium, we can find a mixed-strategy equilibrium G^* such that any point z^* in its support satisfies the first-order condition

$$\hat{z}^* = \frac{u}{2}\int_{\hat{z}^*-\beta}^{\hat{z}^*+\beta} \left(\frac{1}{\beta} - \frac{|\hat{z}^* - z|}{\beta^2}\right) dG^*(z) = \frac{u}{2}\int_{-\infty}^{\infty} \max\left\{0, \frac{1}{\beta} - \frac{|\hat{z}^* - z|}{\beta^2}\right\} dG^*(z).$$

Because $\frac{u}{2\hat{z}^*}\int_{-\infty}^{\infty} \max\left\{0, \frac{1}{\beta} - \frac{|\hat{z}^* - z|}{\beta^2}\right\} dG^*(z) - 1$ is quasiconcave, there exist $\underline{z}$ and $\overline{z}$, $0 \le \underline{z} < \overline{z}$, such that $G^*(\underline{z}) = 0$, $G^*(\overline{z}) = 1$, and $\hat{z}^* - \frac{u}{2}\int_{\hat{z}^*-\beta}^{\hat{z}^*+\beta} \left(\frac{1}{\beta} - \frac{|\hat{z}^* - z|}{\beta^2}\right) dG^*(z) = 0$ for any $\hat{z}^* \in [\underline{z}, \overline{z}]$. Thus, the derivative of $\hat{z}^* - \frac{u}{2}\int_{\hat{z}^*-\beta}^{\hat{z}^*+\beta} \left(\frac{1}{\beta} - \frac{|\hat{z}^* - z|}{\beta^2}\right) dG^*(z)$ must be zero in $[\underline{z}, \overline{z}]$ (i.e.,

$$1 + \frac{u}{2\beta^2}[2G^*(\hat{z}^*) - G^*(\hat{z}^* - \beta) - G^*(\hat{z}^* + \beta)] = 0$$

for all $\hat{z}^* \in [\underline{z}, \overline{z}]$). Applying the boundary condition $G^*(z) = 0$ for $z < \underline{z}$, we can obtain

$$G^*(z) = \left(\left\lfloor \frac{z - \underline{z}}{\beta} \right\rfloor + 1\right) G^*\left(z - \left\lfloor \frac{z - \underline{z}}{\beta} \right\rfloor \beta\right) + \frac{1}{2}\left\lfloor \frac{z - \underline{z}}{\beta} \right\rfloor \left(\left\lfloor \frac{z - \underline{z}}{\beta} \right\rfloor + 1\right) \frac{2\beta^2}{u}$$

for $z \in [\underline{z}, \overline{z} + \beta)$. Furthermore, because $G^*(z) = 1$ for $z \in [\overline{z}, \overline{z} + \beta)$ and $G^*(\cdot)$ is nondecreasing, the mixed strategy in the proposition must be the unique solution. □

Proof of Proposition 8. Following the same argument as that in the proof of Proposition 7, we can verify that the strategies in Proposition 8 are indeed equilibria.

For the uniqueness, let G^* be an equilibrium with marginal distributions (G_1^*, G_2^*). Then, G^* must be an optimal solution to $\min_G \left\{\int \gamma(z) dG(z) : G \text{ has marginals } (G_1^*, G_2^*)\right\}$, and hence, z_2^* increases in z_1^* in the support of G^* following the same argument as that in the proof of Lemma A.1. Because the two events have the same reward and the two competitors are homogeneous, there must exist an equilibrium with marginal distributions (G_2^*, G_1^*), and hence, the mixture of the two equilibria is also an equilibrium as Problems (7)–(8) are a convex program. Thus, if (z_1^*, z_2^*) is in the support of this mixture, so is (z_2^*, z_1^*). Given the monotonicity between z_1^* and z_2^*, it must hold that $z_2^* = z_1^*$ (i.e., the problem can be reduced to a single-event one, in which case uniqueness is guaranteed by Proposition 7). □

Proof of Proposition 9. We verify that G_k^* maximizes class k contestants' objective function in each case.

1. Given other players' strategies, the objective of the class 1 contestant is given by

$$\begin{aligned} &\sum_{j=1,2} \prod_{n' \neq n} G_{n'j}^*(z_j) - \frac{1}{c^2 - \delta_1^2}(cz_1^2 + cz_2^2 - 2\delta_1 z_1 z_2) \\ &\quad = \sum_{j=1,2} \frac{\min\{1, \max^2\{z_j, 0\} + u(\delta_1 - \delta_2)\}}{c + \delta_1} - \frac{1}{c^2 - \delta_1^2}(cz_1^2 + cz_2^2 - 2\delta_1 z_1 z_2), \\ &\quad = -\frac{\delta(z_1 - z_2)^2}{c^2 - \delta_1^2} - \sum_{j=1,2} \frac{z_j^2 - \min\{1, \max^2\{z_j, 0\} + u(\delta_1 - \delta_2)\}}{c + \delta_1} \end{aligned}$$

and class 2 contestants' objective is

$$\sum_{j=1,2} \frac{\min\{1, \max^2\{z_j, 0\}\}}{c + \delta_2} - \frac{z_1^2 + z_2^2 - 2\delta z_1 z_2}{c^2 - \delta_2^2} = -\frac{\delta(z_1 - z_2)^2}{c^2 - \delta_2^2} - \sum_{j=1,2} \frac{z_j^2 - \min\{1, \max^2\{z_j, 0\}\}}{c + \delta_2}.$$

All of the points in $\{(z_1^*, z_2^*) : z_1^* = z_2^* \in [0, \sqrt{u(c+\delta_2)}]\}$ maximize objective functions of the class 1 and class 2 contestants, whereas zero is the unique maximizer of class k, $k > 2$, contestants' objective function

$$\sum_{j=1,2} \frac{\min\{1, \max^2\{z_j, 0\}\}}{c+\delta_2}\left[\frac{\min\{1, \max^2\{0, z_j\} + u(\delta_1 - \delta_2)\}}{u(c+\delta_1)}\right]^{\frac{1}{n_2}} - \frac{z_1^2 + z_2^2 - 2\delta z_1 z_2}{c^2 - \delta_k^2}.$$

Therefore, the strategy in part 1 of the proposition is an equilibrium.

2. Given other players' strategies, the objective function of class 1 contestants

$$\sum_{j=1,2} \frac{\min\{1, \max^2\{z_j, 0\}\}}{c+\delta_1} - \frac{z_1^2 + z_2^2 - 2\delta z_1 z_2}{c^2 - \delta_1^2} = -\frac{\delta(z_1 - z_2)^2}{c^2 - \delta_1^2} - \sum_{j=1,2} \frac{z_j^2 - \min\{1, \max^2\{z_j, 0\}\}}{c+\delta_1}$$

is maximized at all points in $\{(z_1^*, z_2^*) : z_1^* = z_2^* \in [0, \sqrt{u(c+\delta_1)}]\}$, the support of class 1 contestants' strategy, whereas the objective function of class k, $k \geq 2$, contestants

$$\sum_{j=1,2} \left[\frac{\min\{1, \max^2\{z_j, 0\}\}}{c+\delta_1}\right]^{\frac{n_1}{n_1 - 1}} - \frac{z_1^2 + z_2^2 - 2\delta z_1 z_2}{c^2 - \delta_1^2}$$

is maximized at zero. Therefore, the strategy in part 2 of the proposition is an equilibrium. □

Proof of Proposition 10. Let $g(x) = u\int_0^x (c+\delta) dQ^{N-1}(\delta)$. Because Q is a continuous distribution, $g(x)$ is strictly increasing on the support of Q. Then, a competitor's objective function with a realized correlation $\boldsymbol{w}_1^T \boldsymbol{D}_n^{-1} \boldsymbol{w}_2 = \delta$ is

$$uQ^{N-1}(g^{-1}(z_1)^2) + uQ^{N-1}(g^{-1}(z_2)^2) - \frac{z_1^2 + z_2^2 - 2\delta z_1 z_2}{c^2 - \delta^2},$$

which has a unique maximizer $z_1^* = z_2^* = \sqrt{ug(\delta)}$ by the first-order optimality condition. Thus, the pure strategy in the proposition is an equilibrium. □

Proof of Proposition 12. Because $(\tilde{\boldsymbol{w}}_1^{(k)}, \ldots, \tilde{\boldsymbol{w}}_J^{(k)})$ are linearly independent, following a similar argument as in the proof of Lemma A.1,

$$\sum_{n\in\mathcal{N}} \left\{ E_{G_n^{k*}}[\gamma_n(\boldsymbol{z}_n^{k*})] + \max_{\boldsymbol{x}\in\mathbb{R}^M, \boldsymbol{y}\in\mathbb{R}^J} \left\{ \sum_{j=1}^J u_j \prod_{\ell\in\mathcal{N}_{-n}} G_{\ell j}^{k*}\left(\boldsymbol{w}_j^T \boldsymbol{x} + \frac{1}{k} y_j\right) - (\boldsymbol{x}^T \boldsymbol{D}_n \boldsymbol{x} + k\boldsymbol{y}^T \boldsymbol{y}) \right\} \right\} = \sum_{j\in\mathcal{J}} u_j.$$

Let $(\boldsymbol{x}_n^{k*}, \boldsymbol{y}_n^{k*})$ be a maximizer of the above problem such that $(\boldsymbol{x}_n^{k*})^T \boldsymbol{D}_n \boldsymbol{x}_n^{k*} + k(\boldsymbol{y}_n^{k*})^T \boldsymbol{y}_n^{k*} \leq \sum_{j=1}^T u_j$. Because $\max_{\boldsymbol{x}\in\mathbb{R}^M, \boldsymbol{y}\in\mathbb{R}^J} \left\{ \sum_{j=1}^J u_j \prod_{\ell\in\mathcal{N}_{-n}} G_{\ell j}^{k*}(\boldsymbol{w}_j^T \boldsymbol{x} + \frac{1}{k} y_j) - (\boldsymbol{x}^T \boldsymbol{D}_n \boldsymbol{x} + k\boldsymbol{y}^T \boldsymbol{y})\right\} \leq \sum_{j=1}^J u_j$, G_n^{k*} is increasing and bounded, $\boldsymbol{x}_n^{k*}$ is bounded, and $\|\boldsymbol{y}_n^{k*}\|_2 \leq \frac{1}{k}$, there exists a subsequence $\{k_r : r = 1, 2, \ldots\} \subseteq \mathbb{N}$ and $(G_n^*, \boldsymbol{x}_n^*)$, $n = 1, \ldots, N$, such that as $r \to \infty$, $G_n^{k_r*} \to G_n^{k*}$, $\boldsymbol{x}_n^{k_r*} \to \boldsymbol{x}_n^*$, $\boldsymbol{y}_n^{k_r*} \to \boldsymbol{0}$, and

$$\sum_{j=1}^J u_j \prod_{\ell\in\mathcal{N}_{-i}} G_{\ell j}^{k_r*}\left(\boldsymbol{w}_j^T \boldsymbol{x}_n^{k_r*} + \frac{1}{k_r} y_{nj}^{k_r*}\right) - (\boldsymbol{x}_n^{k_r*T} \boldsymbol{D}_n \boldsymbol{x}_n^{k_r*} + k_r \boldsymbol{y}_n^{k_r*T} \boldsymbol{y}_n^{k_r*}) = \max_{\boldsymbol{x}\in\mathbb{R}^m, \boldsymbol{y}\in\mathbb{R}^J} \left\{ \sum_{j\in\mathcal{J}} u_j \prod_{\ell\in\mathcal{N}_{-n}} G_{\ell j}^{k_r*}\left(\boldsymbol{w}_j^T \boldsymbol{x} + \frac{1}{k_r} y_j\right) - (\boldsymbol{x}^T \boldsymbol{D}_n \boldsymbol{x} + k_r \boldsymbol{y}^T \boldsymbol{y}) \right\}$$

converges. Furthermore, for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned}
&\sum_{j=1}^J u_j \prod_{\ell\in\mathcal{N}_{-n}} G_{\ell j}^*(\boldsymbol{w}_j^T \boldsymbol{x}_n^*) - (\boldsymbol{x}_n^{*T} \boldsymbol{D}_n \boldsymbol{x}_n^*) \\
&\geq \lim_{r\to\infty} \left\{ \sum_{j=1}^J u_j \prod_{\ell\in\mathcal{N}_{-n}} G_{\ell j}^{k_r*}\left(\boldsymbol{w}_j^T \boldsymbol{x}_n^{k_r*} + \frac{1}{k_r} y_{nj}^{k_r*}\right) - (\boldsymbol{x}_n^{k_r*T} \boldsymbol{D}_n \boldsymbol{x}_n^{k_r*} + k_r \boldsymbol{y}_n^{k_r*T} \boldsymbol{y}_n^{k_r*}) \right\} \\
&\geq \lim_{r\to\infty} \sum_{j=1}^J u_j \prod_{\ell\in\mathcal{N}_{-n}} G_{\ell j}^{k_r*}(\boldsymbol{w}_j^T \boldsymbol{x}) - \boldsymbol{x}^T \boldsymbol{D}_n \boldsymbol{x} \\
&= \sum_{j=1}^J u_j \prod_{\ell\in\mathcal{N}_{-n}} G_{\ell j}^*(\boldsymbol{w}_j^T \boldsymbol{x}) - \boldsymbol{x}^T \boldsymbol{D}_n \boldsymbol{x},
\end{aligned}$$

and that is, $\boldsymbol{x}_n^*$ maximizes $\sum_{j=1}^J u_j \prod_{\ell\in\mathcal{N}_{-n}} G_j^{\ell*}(\boldsymbol{w}_j^T \boldsymbol{x}) - \boldsymbol{x}^T \boldsymbol{D}_n \boldsymbol{x}$; hence,

$$\sum_{n\in\mathcal{N}} E_{G^*}[\gamma_n(\boldsymbol{z}_n^*)] + \max_{\boldsymbol{x}\in\mathbb{R}^M} \left\{ \sum_{j\in\mathcal{J}} u_j \prod_{\ell\in\mathcal{N}_{-n}} G_j^{\ell*}(\boldsymbol{w}_j^T \boldsymbol{x}) - \boldsymbol{x}^T \boldsymbol{D}_n \boldsymbol{x} \right\} = \sum_{j\in\mathcal{J}} u_j.$$

The proposition then follows from a similar argument as in the proof of Lemma A.1. □

Proofs of Propositions 13 and 14. We assume, without loss of generality, that $\boldsymbol{D}=\boldsymbol{I}$. Similar to Lemma A.1, we can show that G^* is an equilibrium if and only if

$$\max\left\{\sum_{j=1}^{J}u_jG_j^{*N-1}(z_j)-\gamma(\boldsymbol{z})\right\}+E_{G^*}[\gamma(\boldsymbol{z}^*)]=\frac{u_1+\cdots+u_N}{N}. \tag{A.6}$$

1. When $\boldsymbol{w}_j^T\boldsymbol{w}_{j'}\ge 0$ for all j and j', let $\Phi(\boldsymbol{y})=\mathrm{diag}(\boldsymbol{y})\boldsymbol{W}^T\boldsymbol{W}\boldsymbol{y}$ be a function from $\mathbb{R}^J$ to $\mathbb{R}^J$. It is obvious that $\Phi(\boldsymbol{y})$ is continuous and closed. If $\mathbb{R}_+^J\not\subseteq\Phi(\mathbb{R}_+^J)$, then there exists a $\boldsymbol{v}>0$ on the boundary of $\Phi(\mathbb{R}_+^J)$. As Φ is closed, there exists $\boldsymbol{y}>0$ such that $\Phi(\boldsymbol{y})=\boldsymbol{v}$ and the Jacobian at $\boldsymbol{y}$,

$$\mathrm{diag}(\boldsymbol{y})\boldsymbol{W}^T\boldsymbol{W}+\mathrm{diag}\left(\sum_{j=1}^{J}\boldsymbol{w}_1^T\boldsymbol{w}_jy_j,\sum_{j=1}^{J}\boldsymbol{w}_2^T\boldsymbol{w}_jy_j,\ldots\sum_{j=1}^{J}\boldsymbol{w}_J^T\boldsymbol{w}_jy_j\right),$$

is positive definite. Thus, $\boldsymbol{v}$ cannot be on the boundary, and $\mathbb{R}_+^J\subseteq\Phi(\mathbb{R}_+^J)$ must hold. As a result, there exists $\hat{\boldsymbol{y}}\in\mathbb{R}_+^J$ such that $\Phi(\hat{\boldsymbol{y}})=\boldsymbol{u}=(u_1,u_2,\ldots,u_J)^T\in\mathbb{R}_+^J$, and hence, $\hat{\boldsymbol{y}}=(\boldsymbol{W}^T\boldsymbol{W})^{-1}\,\mathrm{diag}(\hat{\boldsymbol{y}})^{-1}\boldsymbol{u}$. Let $\boldsymbol{z}^*=\hat{\boldsymbol{z}}U^{\frac{N-1}{2}}$ with distribution G^*, where U follows a uniform distribution on [0, 1] and $\hat{\boldsymbol{z}}=\boldsymbol{W}^T\boldsymbol{W}\hat{\boldsymbol{y}}=\mathrm{diag}(\hat{\boldsymbol{y}})^{-1}\boldsymbol{u}=\left(\frac{u_1}{\hat{y}_1},\frac{u_2}{\hat{y}_2},\ldots,\frac{u_J}{\hat{y}_J}\right)^T$. Then,

$$E_{G^*}[\gamma(\boldsymbol{z}^*)]=E[U^{N-1}]\hat{\boldsymbol{z}}^{*T}(\boldsymbol{W}^T\boldsymbol{W})^{-1}\hat{\boldsymbol{z}}^*=\frac{1}{N}\boldsymbol{u}^T\mathrm{diag}(\hat{\boldsymbol{y}})^{-1}(\boldsymbol{W}^T\boldsymbol{W})^{-1}\mathrm{diag}(\hat{\boldsymbol{y}})^{-1}\boldsymbol{u}=\frac{u_1+\cdots+u_N}{N},$$

and

$$\begin{aligned}
&\sum_{j=1}^{J}u_jG_j^{*n-1}(z_j)-\boldsymbol{z}^T(\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{z}\\
&=\sum_{j=1}^{J}\frac{u_j}{\hat{z}_j^2}[z_j^2-(z_j\wedge 0)^2+\hat{z}_j^2-(\hat{z}_j\vee z_j)^2]-\boldsymbol{z}^T(\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{z}\\
&\le\sum_{j=1}^{J}\frac{u_jz_j^2}{\hat{z}_j^2}-\boldsymbol{z}^T(\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{z}\\
&=\boldsymbol{z}^T[\mathrm{diag}(\hat{\boldsymbol{y}})\mathrm{diag}(\hat{\boldsymbol{z}})^{-1}-(\boldsymbol{W}^T\boldsymbol{W})^{-1}]\boldsymbol{z}\le 0.
\end{aligned}$$

Thus, (A.6) holds, and the problem can be treated as a single-event problem with $\frac{\tilde{\boldsymbol{w}}}{\|\tilde{\boldsymbol{w}}\|}$, where $\tilde{\boldsymbol{w}}=\boldsymbol{W}(\boldsymbol{W}^T\boldsymbol{W})^{-1}\hat{\boldsymbol{z}}$.

2. When $\boldsymbol{w}_j^T\boldsymbol{w}_{j'}\le 0$ for all j and j', letting $\hat{\lambda}=\frac{J^N+(N-1)(J-1)^N-NJ(J-1)^{N-1}}{NJ^N}\sum_{j=1}^{J}u_j$ and $\hat{\boldsymbol{x}}$ be a random variable such that $P(\hat{\boldsymbol{x}}\in\{t\boldsymbol{w}_j\,|\,t\in[0,z_j]\})=1\wedge\left[\left(\frac{J-1}{J}\right)^{N-1}+\frac{z_j^2}{u_j}\right]^{\frac{1}{N-1}}-\frac{J-1}{J}$ for any $z_j>0$, we have

$$\begin{aligned}
E[\hat{\boldsymbol{x}}^T\hat{\boldsymbol{x}}]&=\sum_{j=1}^{J}\int_0^{\sqrt{u_j\left[1-\left(\frac{J-1}{J}\right)^{N-1}\right]}}z_j^2d\left[\left(\frac{J-1}{J}\right)^{N-1}+\frac{z_j^2}{u_j}\right]^{\frac{1}{N-1}}\\
&=\sum_{j=1}^{J}u_j\int_{\frac{J-1}{J}}^{1}\left[t^{N-1}-\left(\frac{J-1}{J}\right)^{N-1}\right]dt\\
&=\left[\frac{1}{N}-\frac{(J-1)^N+N(J-1)^{N-1}}{NJ^J}\right]\sum_{j=1}^{J}u_j.
\end{aligned}$$

Furthermore, because $\boldsymbol{w}_j^T\boldsymbol{w}_{j'}\le 0$ for all $j\ne j'$, $\boldsymbol{w}_j^T\hat{\boldsymbol{x}}>0$ if and only if $\hat{\boldsymbol{x}}\in\{t\boldsymbol{w}_j\,|\,t\ge 0\}$. Thus, $\hat{G}_j(z_j)=P(\boldsymbol{w}_j^T\hat{\boldsymbol{x}}\le z_j)\le 1\wedge\left[\left(\frac{J-1}{J}\right)^{N-1}+\frac{(0\vee z_j)^2}{u_j}\right]^{\frac{1}{N-1}}$, and for all feasible **z**,

$$E[\hat{\boldsymbol{x}}^T\hat{\boldsymbol{x}}]+\max_{\boldsymbol{z}}\left\{\sum_{j=1}^{J}u_j\hat{G}_j^{N-1}(z_j)-\boldsymbol{z}^T(\boldsymbol{W}^T\boldsymbol{W})^{-1}\boldsymbol{z}\right\}\le\left[\frac{1}{N}+\left(\frac{J-1}{J}\right)^N\right]\sum_{j=1}^{J}u_j.$$

Therefore, $\hat{G}$ is an ε-equilibrium for $\varepsilon\ge\left(\frac{J-1}{J}\right)^N\sum_{j=1}^{J}u_j$. $\square$

Proof of Proposition 15. By Proposition 1,

$$P(\max\{z_{11}^*,\ldots,z_{N1}^*\}\le z)=\left(\frac{(1-w_1^Tw_2v)z^2}{u_1(1-(w_1^Tw_2)^2)}\right)^{\frac{N}{N-1}},$$

and hence,

$$E[\max\{z_{11}^*,\ldots,z_{N1}^*\}]=\int zd\left(\frac{(1-w_1^Tw_2v)z^2}{u_1(1-(w_1^Tw_2)^2)}\right)^{\frac{N}{N-1}}=\frac{2N}{3N-1}\sqrt{\frac{u_1(1-(w_1^Tw_2)^2)}{1-w_1^Tw_2v}}.$$

The first-order condition yields that

$$1=\frac{d}{du_1}E[\max\{z_{11}^*,\ldots,z_{N1}^*\}]=\frac{2N\sqrt{1-(w_1^Tw_2)^2}}{3N-1}\left(\frac{1}{2\sqrt{u_1(1-w_1^Tw_2v)}}+\frac{\sqrt{u_1}w_1^Tw_2}{2\sqrt{1-w_1^Tw_2v}^3}\frac{dv}{du_1}\right),$$

where

$$\frac{dv}{du_1}=\frac{1}{2}\left(\frac{2\left[\left(\frac{u_2}{u_1}-1\right)w_1^Tw_2\right]w_1^Tw_2\frac{u_2}{-u_1^2}+\frac{4u_2}{-u_1^2}}{2\sqrt{\left[\left(\frac{u_2}{u_1}-1\right)w_1^Tw_2\right]^2+\frac{4u_2}{u_1}}}-w_1^Tw_2\frac{u_2}{-u_1^2}\right).$$

By symmetry, the proposition holds. □

Appendix B. Numerical Approximation for the Equilibria in Section 4.2

In this section, we elaborate on the numerical study presented in Section 4.2. When the distribution of the random noise ξ_{nj}, $H(\frac{\cdot}{\beta})$, is Gumbel with a standard deviation of $\frac{\beta\pi}{\sqrt{6}}$, Equation (8) leads to

$$\hat{G}_j(z_j)=\int_{z'}\frac{e^{\frac{z_j}{\beta}}}{e^{\frac{z_j}{\beta}}+e^{\frac{z_j'}{\beta}}}dG_j(z_j'),\ j=1,2,$$

for some G. Additionally, because $E[\gamma(z+\xi_1-\xi_2)]=E[\gamma(z)]+\frac{E[(\xi_1-\xi_2)^T(\xi_1-\xi_2)]}{w_1^Tw_2}$, we can reformulate (7) as a minimization problem with decision variable G:

$$\min_G\int_z\gamma(z)dG(z)+\max_z\left\{\int_{z'}\left(\frac{u_1e^{\frac{z_1}{\beta}}}{e^{\frac{z_1}{\beta}}+e^{\frac{z_1'}{\beta}}}+\frac{u_2e^{\frac{z_2}{\beta}}}{e^{\frac{z_2}{\beta}}+e^{\frac{z_2'}{\beta}}}\right)dG(z')-\gamma(z)\right\},\tag{B.1}$$

where the optimal objective value is $\frac{u_1+u_2}{2}$. It is clear that the support of G^* is contained within $[-2\sqrt{u_1},2\sqrt{u_1}]\times[-2\sqrt{u_2},2\sqrt{u_2}]$ because no contestant will exert an effort that costs more than the reward. For any finite set $\Omega\subset[-2\sqrt{u_1},2\sqrt{u_1}]\times[-2\sqrt{u_2},2\sqrt{u_2}]$, we derive a discrete version of (B.1) with the support of G restricted in Ω:

$$\begin{aligned}\min_{p_z\ge0:\,z\in\Omega}\ &\sum_{z\in\Omega}\gamma(z)p_z+\max_{z\in\Omega}\left\{\sum_{z'\in\Omega}\left(\frac{u_1e^{\frac{z_1}{\beta}}}{e^{\frac{z_1}{\beta}}+e^{\frac{z_1'}{\beta}}}+\frac{u_2e^{\frac{z_2}{\beta}}}{e^{\frac{z_2}{\beta}}+e^{\frac{z_2'}{\beta}}}\right)p_{z'}-\gamma(z)\right\}\\ \text{s.t.}\ &\sum_{z\in\Omega}p_z=1.\end{aligned}$$

This is a linear programming problem with decision variables $\{p_z:z\in\Omega\}$ and the optimal objective value $\frac{u_1+u_2}{2}$ following the same argument as in the proof of Lemma A.1. The equilibrium solution of the discrete version for a specific Ω is given by $G_\Omega^*(z_1,z_2)=\sum_{z\in\Omega\cap(\infty,z_1]\times(\infty,z_2]}p_z^*$.

Let $\Omega_1\subset\Omega_2\subset\cdots$ be a sequence of finite discrete sets such that $\Omega_\infty:=\cup_{k=1}^\infty\Omega_k$ is dense in $[-2\sqrt{u_1},2\sqrt{u_1}]\times[-2\sqrt{u_2},2\sqrt{u_2}]$. For instance, we can define $\Omega_k=\left\{\frac{2i\sqrt{u_1}}{2^k}:i=-2^k,\ldots,2^k\right\}\times\left\{\frac{2i\sqrt{u_2}}{2^k}:i=-2^k,\ldots,2^k\right\}$. Because $G_{\Omega_k}^*$ is bounded and nondecreasing and because the objective function of the inner maximization is continuous in z, there exists a subsequence of $\{G_{\Omega_k}:k=1,2,\ldots\}$ that converges to a distribution function G_{Ω_∞}. Hence, we have $\frac{u_1+u_2}{2}=\int_z\gamma(z)dG_{\Omega_\infty}(z)+\max_{z\in\Omega_\infty}\left\{\int_{z'}\left(\frac{u_1e^{\frac{z_1}{\beta}}}{e^{\frac{z_1}{\beta}}+e^{\frac{z_1'}{\beta}}}+\frac{u_2e^{\frac{z_2}{\beta}}}{e^{\frac{z_2}{\beta}}+e^{\frac{z_2'}{\beta}}}\right)dG_{\Omega_\infty}(z')-\gamma(z)\right\}$. Moreover, because Ω_∞ is dense, the objective in (B.1) achieves its optimal value of $\frac{u_1+u_2}{2}$ at G_{Ω_∞}. Thus, G_{Ω_∞} must be an equilibrium for the continuous version and can be closely approximated with appropriately chosen Ω.

References

[1] Ahmadinejad A, Dehghani S, Hajiaghayi M, Lucier B, Mahini H, Seddighin S (2019) From duels to battlefields: Computing equilibria of blotto and other games. *Math. Oper. Res.* 44(4):1304–1325.
[2] Ales L, Cho SH, Körpeoğlu E (2017) Optimal award scheme in innovation tournaments. *Oper. Res.* 65(3):693–702.
[3] Alpern S, Howard J (2017) Winner-take-all games: The strategic optimisation of rank. *Oper. Res.* 65(5):1165–1176.
[4] Anderson E (2012) Ranking games and gambling: When to quit when you're ahead. *Oper. Res.* 60(5):1229–1244.
[5] Ashlagi I, Monderer D, Tennenholtz M (2011) Simultaneous ad auctions. *Math. Oper. Res.* 36(1):1–13.
[6] Bell RM, Cover TM (1980) Competitive optimality of logarithmic investment. *Math. Oper. Res.* 5(2):161–166.
[7] Boix-Adserà E, Edelman BL, Jayanti S (2020) The multiplayer Colonel Blotto game. *EC'20 Proc. 21st ACM Conf. Econom. Comput.* (ACM, New York), 47–48.
[8] Borel E (1953) The theory of play and integral equations with skew symmetric kernels. *Econometrica* 21(1):97–100.
[9] Corchón LC, Serena M (2018) Contest theory. Corchón LC, Marini MA, eds. *Handbook of Game Theory and Industrial Organization*, vol. II (Edward Elgar Publishing, Cheltenham, UK), 125–146.
[10] DiPalantino D, Vojnovic M (2009) Crowdsourcing and all-pay auctions. *Proc. 10th ACM Conf. Electronic Commerce* (Association for Computing Machinery, New York), 119–128.
[11] Dziubiński M (2013) Non-symmetric discrete general lotto games. *Internat. J. Game Theory* 42(4):801–833.
[12] Grossmann M (2021) Entry regulations and optimal prize allocation in parallel contests. *Rev. Econom. Design* 25(4):289–316.
[13] Hart S (2008) Discrete Colonel Blotto and general lotto games. *Internat. J. Game Theory* 36(3–4):441–460.
[14] Hart S (2016) Allocation games with caps: From Captain Lotto to all-pay auctions. *Internat. J. Game Theory* 45(1–2):37–61.
[15] Hu M, Wang L (2021) Joint vs. separate crowdsourcing contests. *Management Sci.* 67(5):2711–2728.
[16] Ince EL (1956) *Ordinary Differential Equations* (Dover Publications, New York).
[17] Körpeoğlu E, Cho SH (2018) Incentives in contests with heterogeneous solvers. *Management Sci.* 64(6):2709–2715.
[18] Körpeoğlu E, Korpeoglu CG, Hafalır İE (2022) Parallel innovation contests. *Oper. Res.* 70(3):1506–1530.
[19] Kovenock D, Roberson B (2012) Conflicts with multiple battlefields. CESifo Working Paper No. 3165, CESifo, Munich, Germany.
[20] Kovenock D, Roberson B (2021) Generalizations of the General Lotto and Colonel Blotto games. *Econom. Theory* 71:997–1032.
[21] Macdonell ST, Mastronardi N (2015) Waging simple wars: A complete characterization of two-battlefield Blotto equilibria. *Econom. Theory* 58(1):183–216.
[22] Mihm J, Schlapp J (2019) Sourcing innovation: On feedback in contests. *Management Sci.* 65(2):559–576.
[23] Mitroi F-C, Niculescu CP (2011) An extension of Young's inequality. *Abstract Appl. Anal.* 2011:162049.
[24] Myerson RB (1993) Incentives to cultivate favored minorities under alternative electoral systems. *Amer. Political Sci. Rev.* 87(4):856–869.
[25] Rinott Y, Scarsini M, Yu Y (2012) A Colonel Blotto gladiator game. *Math. Oper. Res.* 37(4):574–590.
[26] Roberson B (2006) The Colonel Blotto game. *Econom. Theory* 29(1):1–24.
[27] Shubik M, Weber RJ (1981) Systems defense games: Colonel Blotto, command and control. *Naval Res. Logist. Quart.* 28(2):281–287.
[28] Stouras KI, Erat S, Lichtendahl KC Jr (2025) Dueling contests and platform's coordinating role. *Management Sci.* 71(2):1488–1503.
[29] Stouras KI, Hutchison-Krupat J, Chao RO (2022) The role of participation in innovation contests. *Management Sci.* 68(6):4135–4150.
[30] Terwiesch C, Xu Y (2008) Innovation contests, open innovation, and multiagent problem solving. *Management Sci.* 54(9):1529–1543.
[31] Thomas C (2012) *N*-dimensional Colonel Blotto game with asymmetric battlefield values. Working paper, University of Texas at Austin, Austin.