Resource Allocation in Correlated Competitions

Jiahao He, Jiheng Zhang, Rachel Q. Zhang
Department of Industrial Engineering and Decision Analytics
The Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong S.A.R., China
jhebe@connect.ust.hk, {jiheng, rzhang}@ust.hk

Consider multiple competitors engaging in multiple competitions, each with some *attributes* and a *reward* to the winner or winners if there is a tie. Since the competitions may share certain attributes, a costly effort to improve an attribute may have different effects on a competitor's winning chances in multiple competitions, i.e., the competitions may be *correlated*. Furthermore, such impacts may vary for different competitors due to their abilities in the attributes. We first define the competitor-specific correlation of the competitions and model a competitor's problem as finding a resource allocation to all the attributes that maximizes her expected total reward from all competitions, given other competitors' decisions. We then characterize a symmetric equilibrium decision with two competitions and homogeneous competitors, which can be extended to multiple pair-wise positively or negatively correlated competitions, and asymmetric equilibrium decisions for some special cases with two types of competitors.

Key words: Decision analysis, Noncooperative, Distributions

---

1. Introduction

Competitions are ubiquitous in the fields of politics, economics and sports. In many competitions, only the best performer (or performers if there is a tie) wins a trophy and/or an award. Thus, the goal of a competitor is to outperform her opponents rather than maximize her performance or score. For instance, a politician only needs to win the highest number of votes for a government position, the mutual fund manager with the highest annual return wins the largest share of future investments, and the fastest runner wins a foot race regardless of the winning margins. Each competition requires certain skills or has certain attributes that a competitor can acquire by exerting costly effort and consuming limited resources. Competitors with higher skill levels have a higher probability of winning, although the actual outcome is random and is known at the end of a competition, e.g., after a race is completed. Both internal (e.g., mental and health) and external (e.g., weather, economic, and political) conditions during a competition can contribute to the actual success or failure of a competitor.

In reality, people often need to engage in multiple competitions or events. An administrator may compete to chair several functional committees in an organization; a fashion designer may need to design outfits for sales outlets with different consumer bases; and a gymnast must be prepared for floor, vault, uneven bars, and beam events. Different events may require different or even opposite skills. For instance, artistic training is essential in floor exercises but not critical in vault events, and showmanship may benefit a politician for certain positions but hurt her for others. Thus, events may be positively or negatively correlated. Furthermore, the extent to which a pair of competitions is correlated may be different for various competitors due to their abilities in the attributes. Thus, a competitor's decision is to allocate her limited resources among the *attributes* wisely in order to maximize her expected total reward considering of her own abilities.

If the events do not share any attributes, as implicitly assumed in all existing literature, the attributes do not have to be considered explicitly and a competitor's decision is to simply allocate her resources among the *events* rather than the attributes. Thus, the reward alone is sufficient to describe an event and a competitor's resource allocation decision has a lower dimension as there are usually more attributes across the events than the number of events. A high effort in an event implies a high expected performance in the event, with competitors' equilibrium decisions having been successfully characterized by Roberson (2006) in the case of two competitors whose decisions are continuous variables.

We consider multiple competitors engaging in possibly *correlated* events. We say that two events are correlated if investing in an attribute to improve a competitor's chance of winning one event will have a positive or negative effect on the other. We describe each event by its reward as well as an attribute vector whose elements measure the importance of the attributes for winning the event, and define mathematically the competitor-specific correlation between two events to allow heterogeneity of competitors' abilities. A competitor's decision is to allocate her limited resources among all the attributes to maximize her expected reward from all the events. The intricate relationship between the rewards and attributes of the events and the randomness of the outcomes make competitors' equilibrium behavior much more complex. The difficulty lies in the fact that the direction of improvement in the attributes is not a simple combination of the attribute vectors, in general, especially when some events are negatively correlated. Furthermore, to seek more fundamental understanding of the problem, we will model an equilibrium decision as a set of distributions with unknown and potentially high dimensional supports. Thus, it is a very challenging problem and requires very different methodologies as those in the relevant literature. Our main contributions can be summarized as follows:

1. **Modelling**

(a) **Competitor-specific correlation** We describe the correlation of a pair of events for a competitor as a quadratic function of their attribute vectors with each coefficient in the function indicating a competitor's ability to improve a specific attribute or pair of attributes. Two events are positively (negatively) correlated for a competitor if the value of the function is positive (negative). Thus, two events may be correlated for one competitor but uncorrelated for another and two events with common attributes may be uncorrelated as we will show later.

(b) **Competitors' decision** We can model a competitor's decision as the resources devoted to all the attributes directly. However, doing so requires making a non-trivial assumption that her probability of winning an event is a known function of the resource allocation decisions by all competitors. To avoid the need to identify the winning probability functions a priori and without any knowledge about the structure of the equilibrium decisions, we model a competitor's decision as selecting a distribution of her state in the attributes, which includes a *function* and its *support* as in Alpern and Howard (2017). That is, we aim to discover, rather than make assumptions about, the distribution functions of random variables and their supports. Thus, our model seeks a more fundamental understanding of the problem and is *significantly* more challenging.

2. **Methodology**

(a) **Dimension reduction** Since there can be many attributes across all the events, a competitor's decision can be high dimensional. We are able to transform a competitor's decision from the distributions of the state in the *attributes* to the scores in the *events*. That is, even in the presence of event correlations, we are able to aggregate the attribute information in each event while preserving the information about event correlations for each competitor so that the dimension of her decision is bounded from above by the number of events. In an extreme case where all the events are identical, a competitor's decision is characterized as one-dimension, although the events may have many attributes.

(b) **Converting a zero-sum game into a single optimization** Competitors' equilibrium decision is a solution of a complex zero-sum game and finding it requires solving a set of optimization problems simultaneously. By exploring some properties of the game, we are able to convert it into a single non-convex optimization problem. Although it is still a challenging problem, we succeeded in characterizing the equilibrium decisions under the following scenarios.

3. **Two-event cases**

(a) **Homogeneous competitors** We show the existence of a symmetric equilibrium when the competitors are homogeneous. The equilibrium is unique except in the following two rare cases:

i. The two events are uncorrelated for all competitors, in which case the equilibrium marginal distributions are unique and any distribution with the marginals is an equilibrium.

ii. Two competitors compete in two completely "opposite" events yet with the same reward, in which case any feasible strategy is an equilibrium.

Furthermore, when the two events are positively correlated, the problem is reduced to a single-event one.

(b) **Two types of competitors** If the competitors can be divided into two types, we are able to identify the equilibrium solution when the two events are uncorrelated for both types of competitors. In equilibrium, at least one type of competitors will exert all their effort into the event of their advantage. For general cases, we identify necessary and sufficient conditions under which the problem can be reduced to a single-event one. We also develop an algorithm to compute and analyze the equilibrium behavior when there is a single competitor in each type.

4. **Multiple events and homogeneous competitors** When the events are pair-wise positively correlated, the problem can be reduced to a single-event one with an attribute vector as a linear combination of the attribute vectors of all the events. When the events are pair-wise negatively correlated, we construct an approximate strategy that works well when the number of competitors becomes large. These results suggest efficient heuristic strategies for problems with general events and homogeneous competitors.

The paper is organized as follows. After a brief overview of the relevant literature on competitions in Section 2, we provide a general zero-sum game model and convert it into an equivalent optimization problem in Section 3. We focus on the equilibrium decisions when there are two competitions in Section 4 and extend some results to multiple events in Section 5. The paper concludes in Section 6. All proofs can be found in the Electronic Companion.

2. Literature Review

We first review relevant literature on single-event competitions, referred to as the *contest* literature, and then on multiple-event competitions, referred to as the *resource allocation game* literature.

2.1. Single-event Contests

In a typical contest, a number of competitors decide the amount of costly effort they would exert, which jointly determines the winning probabilities of all competitors through some known *contest success functions*. The winner of the contest receives an award. Thus, a contest can be modelled as a game where each competitor's payoff is a function of their expected reward minus the cost of their effort. Corchón and Serena (2018) provide a comprehensive survey of the contest literature on the justifications and equilibrium analyses under various contest success functions, the design of contest success functions, as well as extended models that allow multiple rounds of a contest, information asymmetry and competitors that are groups instead of individuals.

Although articles in the aforementioned contest literature model a competitor's decision as their effort which determines the probability of winning through a contest success function, there is another line of work that models a competitor's decision as the distribution of the *score* a competitor will receive and the competitor with the highest realized score wins the contest. Instead of a cost function, a constraint is imposed on a competitor's expected score. Bell and Thomas (1980) are the first to study the equilibrium decision in an investment competition under this model, followed by Myerson (1993) who studies a similar game in the context of elections. Anderson (2012) shows that an equilibrium of a multi-stage competition among mutual fund managers can be characterized by an equilibrium score distribution at the last stage. Alpern and Howard (2017) provide a more general constraint on the score distribution and define such a problem as a *distribution ranking game*. Due to the flexibility of the generalized constraint, they are able to provide an innovative method to derive the equilibrium decision for the multi-player silent duel game under this model.

2.2. Multiple-event Contests

When there is more than one event, the problem changes fundamentally. A competitor's decision is to allocate a limited amount of effort or *resources* among the events to maximize her expected total number of winning events and each event is won by the competitor with the highest level of resources. Such a game is referred to as a Colonel Blotto game introduced by Borel (1953) where two competitors compete in multiple contests or *battlefields*. Ahmadinejad et al. (2019) develop a polynomial-time algorithm to compute an equilibrium.

However, when the resources are infinitely divisible, Roberson (2006) is able to fully characterize the equilibrium solution for the game with a general number of battlefields. Since then, researchers have identified equilibrium decisions for several variants of the basic Colonel Blotto game, focusing mostly on the case with two competitors. Some modify the budget constraints in the game, e.g., Macdonell and Mastronardi (2015) allow non-linear resource constraints in a two-event contest, while Hart (2008), Kovenock and Roberson (2010), Dziubiński (2013), Hart (2016) and Kovenock and Roberson (2020) allow the budgets to be constrained in expectation. Others consider more complex objective functions. Thomas (2012) constructs equilibrium solutions for contests with heterogeneous rewards and illustrate them through the election of US presidents. Shubik and Weber (1981) introduce complementarity among the battlefields, so the goal is not to win as many battlefields as one can but to win a bundle of important battlefields, Rinott et al. (2012) study a Colonel Blotto gladiator game where resources (power) need to be allocated to the gladiators in a team and the surviving gladiators from two teams compete in a series of events. Whichever team is still standing at the end wins the game. To our knowledge, Boix-Adserà et al. (2020) is the only paper that constructs efficiently-sampleable symmetric equilibria for cases with multiple identical competitors.

In all of the above games, events are either identical or differ only by theirs rewards, and a competitor's decision is to allocate her resources to all the events. In reality, each event may require multiple skills or have multiple attributes, and an attribute may be shared by multiple events, leading to possible correlations among the events. Thus, a competitor's decision is much more complex, and we will model event correlations and study competitor equilibrium decisions.

3. Problem Formulation and General Properties

Consider n competitors, indexed by $i \in \mathcal{I} = \{1, \cdots, n\}$, competing in J events, indexed by $j \in \mathcal{J} = \{1, \cdots, J\}$, with a total of m attributes, some shared by multiple events and others unique to an event.

1. **Events.** We describe event j by its reward $u_j > 0$ and an attribute vector $\mathbf{w}_j \in \mathbb{R}^m$, measuring the importance of the m attributes to the winning chance of event j. The higher a positive element in $\mathbf{w}_j$ is, the more critical the corresponding attribute is. A negative (zero) value implies that a particular attribute contributes negatively (has no impact) on winning event j. Without loss of generality, we restrict $\mathbf{w}_j$ to be a unit vector and hence, $-1 \le \mathbf{w}_j^T \mathbf{w}_{j'} \le 1$ for any $j, j' \in \mathcal{J}$. The following equivalences will facilitate our analysis:

- $\mathbf{w}_j^T \mathbf{w}_{j'} = 1 \Leftrightarrow \mathbf{w}_j = \mathbf{w}_{j'}$, i.e., j and j' are identical events;
- $\mathbf{w}_j^T \mathbf{w}_{j'} = -1 \Leftrightarrow \mathbf{w}_j = -\mathbf{w}_{j'}$, i.e., j and j' are completely opposite events;
- $-1 < \mathbf{w}_j^T \mathbf{w}_{j'} < 1 \Leftrightarrow \mathbf{w}_j$ and $\mathbf{w}_{j'}$ are linearly independent.

2. **Competitor's decision and reward** We define competitor i's decision as selecting a distribution function $F^i(\cdot)$ of her state in the attributes, $\mathbf{X}_i \in \mathbb{R}^m$, as a result of her resource allocation decision. Since some elements of $\mathbf{w}_j$ can be negative, we allow $\mathbf{X}_i$ to be a real vector and denote $\mathbf{x}_i$ as its realization. Furthermore, we assume that all the competitors make their decisions simultaneously to maximize their expected total rewards, and hence $\mathbf{X}_1, \cdots, \mathbf{X}_n$ are independent random vectors.

Competitor i's weighted state $\mathbf{w}_j^T \mathbf{X}_i$ can be understood as her score in event j. The competitor or competitors with the highest realized score win event j, regardless of their winning margin, and share the reward u_j if there is a tie. Thus, restricting $\|\mathbf{w}_j\| = 1$ indeed involves no loss of generality.

3. **Competitor's resource constraint.** As a high state $\mathbf{X}_i$ requires more resources which are limited, we impose the constraint that $E_{F^i}[\mathbf{X}_i^T \mathbf{D}_i \mathbf{X}_i] \le 1$ where $\mathbf{D}_i \in \mathbb{R}^{m \times m}$ has the following properties.

- We require that $\mathbf{D}_i$ be positive definite to exclude the possibility of a score to be infinite, e.g., $\mathbf{w}_j^T \mathbf{X}_i = \infty$, under a feasible decision. Thus, the diagonal elements are all positive and, the higher a diagonal element, the more difficult it is (more resources are needed) for competitor i to improve the corresponding attribute. A positive (negative) off-diagonal element indicates that extra (less) effort is needed if a competitor tries to improve a pair of attributes simultaneously due to their dissimilarity (similarity). Thus, $\mathbf{D}_i$ is a comprehensive description of competitor i's capabilities.
- It is easy to see that $\mathbf{D}_i^{-1} \mathbf{w}_j$ is the direction along which competitor i gains the greatest chance to win event j and $\sqrt{\mathbf{w}_j^T \mathbf{D}_i^{-1} \mathbf{w}_j}$ is the highest expected score she can achieve in event j. Thus, $\mathbf{w}_j^T \mathbf{D}_i^{-1} \mathbf{w}_j$ measures the overall competitiveness of competitor i in event j and, the higher the value, the stronger competitor i is in event j.

4. **Competitor-specific event correlations.** When competitor i tries to improve her score in event j along $\mathbf{D}_i^{-1}\mathbf{w}_j$, the impact on her score in event j', $j' \neq j$, depends on the angle between $\mathbf{D}_i^{-1}\mathbf{w}_j$ and $\mathbf{w}_{j'}$. Thus, we define the correlation between events j and j' for competitor i as $\mathbf{w}_j^T\mathbf{D}_i^{-1}\mathbf{w}_{j'}$ which is the cosine of the angle between $\mathbf{D}_i^{-1}\mathbf{w}_j$ and $\mathbf{w}_{j'}$ if $\mathbf{D}_i^{-1}\mathbf{w}_j$ is also a unit vector. It may differ for different competitors and is different from the concept of correlation between two random variables.

For competitor i, events j and j' are positively (negatively) correlated if $\mathbf{w}_j^T\mathbf{D}_i^{-1}\mathbf{w}_{j'} > 0$ (< 0) and uncorrelated if $\mathbf{w}_j^T\mathbf{D}_i^{-1}\mathbf{w}_{j'} = 0$ in which case improving the winning chance in one event has no effect on that in the other event. Thus, two events can be uncorrelated for a competitor even if they share certain attributes, e.g., when $\mathbf{D}_i^{-1}\mathbf{w}_j = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\mathbf{w}_{j'} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Throughout the paper, we use the superscript " $*$ " to represent equilibrium functions or values, and boldfaced letters to represent vectors or matrices whose dimensions will be clear from the context. We define an *equilibrium* decision as a tuple $(F^{1*}, \ldots, F^{n*})$ such that F^{i*} of $\mathbf{X}_i^*$ is the solution to the following linear program at $\mathbf{F}^{-i*} = \left\{F^{1*}, \ldots, F^{(i-1)*}, F^{(i+1)*}, \ldots, F^{n*}\right\}$ of $(\mathbf{X}_1^*, \cdots, \mathbf{X}_{i-1}^*, \mathbf{X}_{i+1}^*, \cdots, \mathbf{X}_n^*)$,

$$\max_{F^i} \sum_{j\in\mathcal{J}} u_j \sum_{N\subseteq\mathcal{I}_{-i}} \frac{1}{|N|+1} P_{F^i,\mathbf{F}^{-i*}}\left(\mathbf{w}_j^T\mathbf{X}_i = \mathbf{w}_j^T\mathbf{X}_\ell^* > \mathbf{w}_j^T\mathbf{X}_{\ell'}^*,\ \forall(\ell,\ell')\in N\times(\mathcal{I}_{-i}-N)\right) \tag{1}$$

$$\text{s.t.}\quad E_{F^i}\left(\mathbf{X}_i^T\mathbf{D}_i\mathbf{X}_i\right) \le 1, \tag{2}$$

where $\mathcal{I}_{-i}$ is the set of all competitors except competitor i. When competitor i wins event j along with competitors in $N\subseteq\mathcal{I}_{-i}$, which happens with probability $P_{F^i,\mathbf{F}^{-i*}}(\mathbf{w}_j^T\mathbf{X}_i = \mathbf{w}_j^T\mathbf{X}_\ell^* > \mathbf{w}_j^T\mathbf{X}_{\ell'}^*,\ \forall(\ell,\ell')\in N\times(\mathcal{I}_{-i}-N))$, the reward u_j is evenly divided among the $|N|+1$ winners.

Thus, the resource allocation problem is a zero-sum game and finding an equilibrium solution requires solving n complicated optimization problems simultaneously. Below, we will first reformulate the problem as determining the distributions of the scores in Section 3.1, then convert the problem into a single non-convex optimization one in Section 3.2, and further establish that finding an equilibrium solution is equivalent to finding the marginal distributions of the scores in Section 3.3.

3.1. Problem Reformulation and Dimension Reduction

Let $Z_j^i = \mathbf{w}_j^T\mathbf{X}_i$ denote competitor i's score in event j and G_j^i its distribution. Then, $\mathbf{Z}^i \in \operatorname{Im}(\mathbf{W}^T) = \{\mathbf{W}^T\mathbf{x} : \mathbf{x}\in\mathbb{R}^m\} \subset \mathbb{R}^J$ where $\mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_J)$. Finding an equilibrium solution $(F^{1*}, \ldots, F^{n*})$ of the state $(\mathbf{X}_1^*, \ldots, \mathbf{X}_n^*)$ is equivalent to finding distributions $(G^{1*}, \ldots, G^{n*})$ of the scores $(\mathbf{Z}^{1*}, \ldots, \mathbf{Z}^{n*})$ where $\mathbf{Z}^{i*}\in\operatorname{Im}(\mathbf{W}^T)$ such that G^{i*} solves

$$\max_{G^i} \sum_{j\in\mathcal{J}} u_j \sum_{N\subseteq\mathcal{I}_{-i}} \frac{1}{|N|+1} P_{G^i,\mathbf{G}^{-i*}}\left(Z_j^i = Z_j^{\ell*} > Z_j^{\ell'*},\ \forall(\ell,\ell')\in N\times(\mathcal{I}_{-i}-N)\right) \tag{3}$$

$$\text{s.t.}\quad E_{G^i}\left(\Gamma_i(\mathbf{Z}^i)\right) \le 1, \tag{4}$$

where $\Gamma_i(\mathbf{z}) = \min_{\mathbf{x}} \{\mathbf{x}^T \mathbf{D}_i \mathbf{x} : \mathbf{W}^T \mathbf{x} = \mathbf{z}\}$. Thus, we also refer to $(G^{1*}, \ldots, G^{n*})$ as an *equilibrium* from which we can recover F_i^* as $\mathbf{X}_i^* = \arg\min_{\mathbf{x}} \{\mathbf{x}^T \mathbf{D}_i \mathbf{x} : \mathbf{W}^T \mathbf{x} = \mathbf{Z}^{i*}\}$, the minimum efforts to score $\mathbf{Z}^{i*}$.

When there are only two events, i.e., $J = 2$, $\mathbf{W}^T\mathbf{W}$ is non-singular if and only if $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent and

$$\Gamma_i(\mathbf{z}) = \begin{cases} \mathbf{z}^T \hat{\mathbf{D}}_i \mathbf{z}, & \text{if } \mathbf{W}^T\mathbf{W} \text{ is non-singular,} \\ \frac{z_1^2}{\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_1}, & \text{if } \mathbf{W}^T\mathbf{W} \text{ is singular and } \mathbf{z} \in \mathrm{Im}(\mathbf{W}^T), \\ \infty, & \text{otherwise,} \end{cases} \tag{5}$$

$$\mathbf{X}_i^* = \begin{cases} \mathbf{D}_i^{-1} \mathbf{W} \hat{\mathbf{D}}_i \mathbf{Z}^{i*}, & \text{if } \mathbf{W}^T\mathbf{W} \text{ is non-singular,} \\ \frac{\mathbf{D}_i^{-1} \mathbf{w}_1 Z_1^{i*}}{\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_1}, & \text{otherwise,} \end{cases}$$

where

$$\hat{\mathbf{D}}_i = \left(\mathbf{W}^T \mathbf{D}_i^{-1} \mathbf{W}\right)^{-1} = \frac{1}{|\mathbf{W}^T \mathbf{D}_i^{-1} \mathbf{W}|} \begin{pmatrix} \mathbf{w}_2^T \mathbf{D}_i^{-1} \mathbf{w}_2 & -\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_2 \\ -\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_2 & \mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_1 \end{pmatrix}.$$

That is, we can aggregate the m attributes for each competitor and reduce the dimension of the problem from m to J. When $J = 2$, if $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent or equivalently, $-1 < \mathbf{w}_1^T \mathbf{w}_2 < 1$, we can simplify the description of competitor i with a symmetric matrix $\hat{\mathbf{D}}_i = (\mathbf{W}^T \mathbf{D}_i^{-1} \mathbf{W})^{-1} \in \mathbb{R}^{2\times 2}$, rather than $\mathbf{D}_i \in \mathbb{R}^{m\times m}$. The diagonal elements in $\hat{\mathbf{D}}_i$ indicate competitor i's ability to improve the scores in the two events and the single off-diagonal element is exactly the correlation between the two events for competitor i. When $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent, i.e., $\mathbf{w}_1 = \pm \mathbf{w}_2$, $\hat{\mathbf{D}}_i$ is replaced by a scalar $\left(\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_1\right)^{-1} \in \mathbb{R}$ and competitor i's decision is one dimensional.

3.2. An Equivalent Optimization Problem

Note that, even with the reduction of dimensionality, the problem is still distinct from the literature as the events share attributes and may be correlated. Moreover, this is still a zero-sum game whose solution is n distributions with unknown supports and a highly challenging problem. Rather than trying to solve Problem (3)-(4) for all i simultaneously, we establish that a sole winner exists in each event almost surely if there exists $\mathbf{x} \in \mathbb{R}^m$ such that the score vector $\mathbf{W}^T \mathbf{x} > 0$. With that, we are able to convert the zero-sum game into a single optimization problem in Theorem 1 whose proof can be found in the Electronic Companion. Otherwise, we show in Proposition 1 that there can be at most two winners in each event almost surely and solving the single optimization problem is sufficient to find an equilibrium if it exists when $J = 2$. Let $(G_1^{i*}, G_2^{i*}, \ldots, G_J^{i*})$ denote the marginal distributions of G^{i*} $(\mathbf{Z}^{i*})$ and $\mathbf{z}^i$ as the realization of $\mathbf{Z}^i$.

Theorem 1. *Suppose that there exists* $\mathbf{z}^i > 0$.

1. $P_{G^{i*},\mathbf{G}^{-i*}}\left(Z_j^{i*} = Z_j^{i'*} \geq Z_j^{\ell *}, \forall \ell \in \mathcal{I} - \{i,i'\}\right) = 0$ *for any* $i' \in \mathcal{I}_{-i}$*, i.e., there is a sole winner in each event almost surely.*

2. $(G^{1*},\ldots,G^{n*})$ *is an equilibrium if and only if* $E_{G^{i*}}\left(\Gamma_i(\mathbf{Z}^{i*})\right) = 1$ *for all* i *and*

$$\min_{(\lambda_1,\ldots,\lambda_n)\geq \mathbf{0}} \left\{ \sum_{i\in\mathcal{I}} \left\{ \lambda_i + \max_{\mathbf{z}\in \mathrm{Im}(\mathbf{W}^T)} \left\{ \sum_{j\in\mathcal{J}} u_j \prod_{\ell\in\mathcal{I}_{-i}} G_j^{\ell *}(z_j) - \lambda_i \Gamma_i(\mathbf{z}) \right\}\right\}\right\} = \sum_{j\in\mathcal{J}} u_j. \tag{6}$$

A necessary condition for the existence of a positive $\mathbf{z}^i$ is that $\mathbf{w}_j + \mathbf{w}_{j'} \neq 0$ for any two events j and j', i.e., there does not exist two completely opposite events, and two sufficient conditions are $\mathbf{w}_1,\ldots,\mathbf{w}_J$ are linearly independent and $\mathbf{w}_1 = \cdots = \mathbf{w}_J$. When there exists a positive $\mathbf{z}^i$, competitor i can improve her scores in all events and avoid any tie almost surely by shifting the support of $\mathbf{Z}^i$ along $\mathbf{z}^i$. Let $(\lambda_1^*,\lambda_2^*,\ldots,\lambda_n^*)$ be a minimizer of the left hand side of (6). Then, the support of $\mathbf{Z}^{i*}$ is a bounded subset of the maximizers of $\sum_{j\in\mathcal{J}} u_j \prod_{\ell\in\mathcal{I}_{-i}} G_j^{\ell *}(z_j) - \lambda_i^* \Gamma_i(\mathbf{z})$.

By Theorem 1, the resource constraint for each competitor is tight at an equilibrium, i.e., all competitors use up their resources. Finding an equilibrium solution is equivalent to finding a solution that satisfies $E_{G^{i*}}\left(\Gamma_i(\mathbf{Z}^{i*})\right) = 1$ for all $i \in \mathcal{I}$ and (6) which involves a single optimization problem. That is, one can convert a zero-sum game, which requires solving n optimization problems simultaneously, into a single optimization problem.

When $\mathbf{w}_1,\ldots,\mathbf{w}_J$ are linearly dependent, improving the scores in some events may hurt the scores in some other events and the overall impact depends on the rewards and $\mathbf{W}$ in a complicated manner. When $J = 2$ and $\mathbf{w}_1 + \mathbf{w}_2 = 0$, there can be at most two winners in an event almost surely and condition (7), rewritten from condition (6), is a sufficient one.

Proposition 1. *Suppose that* $J = 2$ *and* $\mathbf{w}_1 + \mathbf{w}_2 = 0$.

1. $P_{G^{i*},\mathbf{G}^{-i*}}\left(Z_j^{i*} = Z_j^{i'*} = Z_j^{i''*}\right) = 0$ *for all* $i', i'' \in \mathcal{I}_{-i}$ *and* $i' \neq i''$*, i.e., there are at most two winners in each event almost surely.*

2. *The conditions in Theorem 1 (2) is sufficient, i.e.,* $(G^{1*},\ldots,G^{n*})$ *is an equilibrium if* $E_{G^{i*}}\left[\left(Z_1^{i*}\right)^2\right] = \mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_1$ *for all* i *and*

$$\min_{(\lambda_1,\ldots,\lambda_n)\geq \mathbf{0}} \left\{ \sum_{i\in\mathcal{I}} \left\{ \lambda_i + \max_{z_1} \left\{ u_1 \prod_{\ell\in\mathcal{I}_{-i}} G_1^{\ell *}(z_1) + u_2 \prod_{\ell\in\mathcal{I}_{-i}} (1 - G_1^{\ell *}(z_1)) - \frac{\lambda_i z_1^2}{\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_1} \right\}\right\}\right\} = u_1 + u_2. \tag{7}$$

First we note that, if all the distributions in $\mathbf{G}^{-i}$ are continuous, a single winner almost surely emerges regardless of competitor i's strategy. Theorem 1 and Proposition 1 imply that, in equilibrium, at most one distribution (two distributions) can be discontinuous at any given point when $\mathbf{w}_1 + \mathbf{w}_2 \neq 0$ ($\mathbf{w}_1 + \mathbf{w}_2 = 0$). Suppose that a particular competitor's strategy in $\mathbf{G}^{-i*}$ is discontinuous at a point $\hat{\mathbf{z}}$. Then, competitor i's winning probability in event j as a function of $\mathbf{z} \in \mathrm{Im}(\mathbf{W}^T)$,

$\sum\limits_{N\subseteq\mathcal{I}_{-i}} \frac{1}{|N|+1} P_{\mathbf{G}^{-i*}} \left(Z_j^{\ell*} = z_j > Z_j^{\ell'*},\ \forall(\ell,\ell')\in N\times(\mathcal{I}_{-i}-N)\right)$, is discontinuous at $\hat{\mathbf{z}}$. Her expected reward is also discontinuous when $\mathbf{w}_1+\mathbf{w}_2\neq 0$, while it may be continuous at this point when $\mathbf{w}_1+\mathbf{w}_2=0$ as a positive jump in the winning probability in one event may be cancelled out by the negative effect on that in the other. However, the expected reward would be discontinuous if multiple competitors' strategies in $\mathbf{G}^{-i*}$ are discontinuous at this point when $\mathbf{w}_1+\mathbf{w}_2=0$. A competitor is better off by including only one side of and excluding the discontinuous point in her support. Thus, a sole winner exists in each event when $\mathbf{w}_1+\mathbf{w}_2\neq 0$ and there are at most two winners when $\mathbf{w}_1+\mathbf{w}_2=0$. For instance, when $\mathbf{w}_1+\mathbf{w}_2=0$, $n=2$, and $u_1=u_2$, the deterministic strategy with $\mathbf{Z}^1=\mathbf{Z}^2=(z_1^*,-z_1^*)^T$ is an equilibrium under which each event is won by both competitors for sure. In this case, G_1^i (G_2^i) is discontinuous at z_1^* ($-z_1^*$) while the expected reward is constant over $\mathrm{Im}(\mathbf{W}^T)$. When $u_1>u_2$, both competitors will try to be the sole winner of event 1 and there is a sole winner in each event almost surely.

3.3. Marginals as Decisions

Note that condition (6) only involves the marginal distributions and the joint distributions only appear in the constraints $E_{G^{i*}}(\Gamma_i(\mathbf{Z}^{i*}))=1$. Thus, finding an equilibrium $(G^{1*},\ldots,G^{n*})$ is equivalent to finding marginal distributions $(G_1^{i*},\ldots,G_J^{i*})$ that satisfies (6) and

$$\min_{G^i}\left\{E_{G^i}\left(\Gamma_i(\mathbf{Z}^i)\right)\middle|G^i \text{ has marginal distributions } (G_1^{i*},\ldots,G_J^{i*})\right\}=1 \tag{8}$$

for all $i\in\mathcal{I}$. Thus, an infinite number of equilibria may exist.

Note that the optimization problem in (8) is a *multi-marginal optimal transportation problem* with quadratic costs (see e.g., Pass (2012)), and the general structure of the support of $\mathbf{Z}^{i*}$ is unknown and complicated. Interestingly, when $J=2$, it reduces to a classic transportation problem where the supports of G_1^i and G_2^i are the origins and destinations, respectively. Origin z_1^i has $G_1^i(z_1^i)$ amount of supply and the demand at destination z_2^i is $G_1^i(z_1^i)$, and it costs $\Gamma_i(z_1^i,z_2^i)$ to transport one unit from z_1^i to z_2^i. With that, we first focus on the case with $J=2$.

4. Two Events

We say that a set in $\mathbb{R}^2$ is increasing (decreasing) if, for any two points $\mathbf{x}$ and $\mathbf{y}$ in the set, $(x_1-y_1)(x_2-y_2)\geq 0$ (≤ 0). Proposition 2 provides the relationship between an equilibrium, if one exists, and its marginals, which also reveals the shape of the support of $\mathbf{Z}^{i*}$. When the two events are uncorrelated, Γ_i is separable in z_1 and z_2 and any $(G^1,\cdots,G^n)$ with the marginals (G_1^{i*},G_2^{i*}), $i\in\mathcal{I}$, is an equilibrium. When the two events are positively (negatively) correlated, the support of $\mathbf{Z}^{i*}$ is increasing (decreasing) as expected. These properties are critical in identifying an equilibrium solution.

Proposition 2. *The relationship between an equilibrium, if one exists, and its marginals is as follows:*

1. *If* $\mathbf{w}_1^T\mathbf{D}_i^{-1}\mathbf{w}_2=0$*, any distribution with the marginals* (G_1^{i*},G_2^{i*}) *is an equilibrium.*
2. *If* $\mathbf{w}_1^T\mathbf{D}_i^{-1}\mathbf{w}_2>0$*,* $G^{i*}(\mathbf{z})=G_1^{i*}(z_1)\wedge G_2^{i*}(z_2)$*, i.e.,* $\mathbf{Z}^{i*}$ *has an increasing support.*
3. *If* $\mathbf{w}_1^T\mathbf{D}_i^{-1}\mathbf{w}_2<0$*,* $G^{i*}(\mathbf{z})=[G_1^{i*}(z_1)+G_2^{i*}(z_2)-1]^+$*, i.e.,* $\mathbf{Z}^{i*}$ *has a decreasing support.*

4.1. Homogeneous Competitors

With homogeneous competitors, $\mathbf{D}_i=\mathbf{D}$ for all $i\in\mathcal{I}$. As we can view the problem as one with $\tilde{\mathbf{w}}_j=\sqrt{\mathbf{D}}^{-1}\mathbf{w}_j$, $\tilde{\mathbf{X}}_i=\sqrt{\mathbf{D}}\mathbf{X}_i$, and $E\left(\tilde{\mathbf{X}}_i^T\tilde{\mathbf{X}}_i\right)\le 1$, it is sufficient to analyze the case where $\mathbf{D}=\mathbf{I}$ and the correlation between the two events is simply $\mathbf{w}_1^T\mathbf{w}_2$ for all competitors. The two events are uncorrelated if $\mathbf{w}_1^T\mathbf{w}_2=0$ and positively (negatively) correlated if $\mathbf{w}_1^T\mathbf{w}_2>0$ (<0), respectively. Furthermore, we only consider symmetric equilibria and omit the index i in our notation. Then, condition (6) becomes

$$\lambda^*+\max_{\mathbf{z}}\left\{\sum_{j=1}^{2}u_jG_j^*(z_j)^{n-1}-\lambda^*\Gamma(\mathbf{z})\right\}=\frac{u_1+u_2}{n}$$

where

$$\Gamma(\mathbf{z})=\begin{cases}\frac{z_1^2-2\mathbf{w}_1^T\mathbf{w}_2z_1z_2+z_2^2}{1-(\mathbf{w}_1^T\mathbf{w}_2)^2}, & \text{if } \mathbf{W}^T\mathbf{W} \text{ is non-singular,}\\ z_1^2, & \text{otherwise.}\end{cases}$$

Under symmetric equilibria, Theorem 1 implies that Z_j^* is a continuous random variable or $G_j^*(z_j)$ is continuous in z_j. That is, $G_j^*(z_j)$ should be differentiable almost everywhere. To simplify the analysis, we make the following assumption.

Assumption 1. $G_j^*(z_j)$ *is differentiable in the support of* Z_j^**.*

Under this assumption, condition (6) is also necessary when $\mathbf{w}_1+\mathbf{w}_2=0$ if $n>2$ or $u_1\neq u_2$. Furthermore, by Theorem 1, any maximizer of $\sum_{j=1}^{2}u_j\left[G_j^*(z_j)\right]^{n-1}-\lambda^*\Gamma(\mathbf{z})$ must satisfy the first-order optimality conditions,

$$(n-1)u_j[G_j^*(z_j)]^{n-2}G_j^{*'}(z_j)=\frac{2\lambda^*}{1-(\mathbf{w}_1^T\mathbf{w}_2)^2}\left(z_j-\mathbf{w}_1^T\mathbf{w}_2z_{3-j}\right),\ j=1,2. \tag{9}$$

We show the existence of multiple equilibria when $\mathbf{w}_1^T\mathbf{w}_2=0$ or the two events are uncorrelated in Section 4.1.1. When the two events are correlated, we show the existence and uniqueness of an equilibrium when $0<|\mathbf{w}_1^T\mathbf{w}_2|<1$ in Section 4.1.2 and analyze the case when $\mathbf{w}_1^T\mathbf{w}_2=\pm1$ in Section 4.1.3.

4.1.1. Existence of Multiple Equilibria When $\mathbf{w}_1^T\mathbf{w}_2 = 0$ When the two events are uncorrelated, condition (9) only involves z_j and its distribution function G_j^*. Thus, the marginal distributions can be found separately and an infinite number of equilibria exists as stated in Proposition 3.

PROPOSITION 3. *When $\mathbf{w}_1^T\mathbf{w}_2 = 0$, a two-dimensional distribution G^* is an equilibrium if and only if its marginal distributions are $G_j^*(z_j) = \left(\frac{u_1+u_2}{nu_j} z_j^2\right)^{\frac{1}{n-1}}$ with $E(Z_j^*)^2 = \frac{u_j}{u_1+u_2}$, $j = 1, 2$.*

That is, a competitor can decide the distribution of the scores of the two events separately and the effort devoted to the score in an event is proportional to its reward. For marginals G_1^* and G_2^* given above, $G^*(\mathbf{z}) = G_1^*(z_1)G_2^*(z_2)$ under which Z_1^* and Z_2^* are independent, $G^*(\mathbf{z}) = \min\{G_1^*(z_1), G_2^*(z_2)\}$ under which $\frac{Z_1^*}{Z_2^*} = \left(\frac{u_2}{u_1}\right)^{\frac{n-1}{2}}$, and $G^*(\mathbf{z}) = [G_1^*(z_1) + G_2^*(z_2) - 1]^+$ under which $\left(\frac{Z_1^{*2}}{u_1}\right)^{\frac{1}{n-1}} + \left(\frac{Z_2^{*2}}{u_2}\right)^{\frac{1}{n-1}} = \frac{n}{u_1+u_2}$ are all equilibrium distributions.

Proposition 3, obtained for the case with two uncorrelated events and multiple competitors, is similar to some existing results, e.g., Theorem 1 in Kovenock and Roberson (2020) which characterizes equilibrium resource allocation decisions of two heterogeneous competitors' competing in multiple events without shared attributes.

4.1.2. Existence and Uniqueness of an Equilibrium When $0 < |\mathbf{w}_1^T\mathbf{w}_2| < 1$ In this case, the two events are correlated and $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent but not orthogonal to each other. Thus, there is a one-to-one correspondence between z_1^* and z_2^* by (9) and hence, the support of $\mathbf{Z}^*$ must be one-dimensional. Furthermore, given that $\mathbf{z}^*$ is a maximizer of $\sum_{j=1}^{2} u_j \left[G_j^*(z_j)\right]^{n-1} - \lambda^* \mathbf{z}^T(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}$ by Theorem 1, the support of $\mathbf{Z}^*$ must be a continuous curve, as otherwise there would exist a rectangular region (not in the support) in which G^* is a constant in $(0, 1)$ and a higher objective value than $\sum_{j=1}^{2} u_j \left[G_j^*\left(z_j^*\right)\right]^{n-1} - \lambda^* \mathbf{z}^{*T}(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}^*$ occurs at a certain point.

LEMMA 1. *The support of $\mathbf{Z}^*$ is a continuous and strictly monotonic one-dimensional curve in the cone spanned by $(1, \mathbf{w}_1^T\mathbf{w}_2)^T$ and $(\mathbf{w}_1^T\mathbf{w}_2, 1)^T$, i.e., it is between the lines $z_1 = z_2\mathbf{w}_1^T\mathbf{w}_2$ and $z_2 = z_1\mathbf{w}_1^T\mathbf{w}_2$, and the marginal distribution G_j^* is strictly increasing when $G_j^*(z_j) \in (0, 1)$.*

We refer to the cone spanned by $(1, \mathbf{w}_1^T\mathbf{w}_2)^T$ and $(\mathbf{w}_1^T\mathbf{w}_2, 1)^T$as *the cone* for convenience. By Lemma 1, we can represent the support of $\mathbf{Z}^*$ by $\{\mathbf{z}(t) = (z_1(t), z_2(t))^T : t \in [0, 1]\}$ where $z_1(t)$ is the t-quantile of G_1^* and $z_2(t)$ is uniquely determined by $z_1(t)$. Furthermore, $z_1(t)$ is continuous and strictly increasing in $t \in (0, 1)$ and $z_2(t)$ is strictly monotone and continuous in t. By Proposition 2, $z_2(t)$ is strictly increasing (decreasing) if and only if $\mathbf{w}_1^T\mathbf{w}_2 > 0$ ($\mathbf{w}_1^T\mathbf{w}_2 < 0$). Proposition 4 establishes the existence and uniqueness of an equilibrium.

PROPOSITION 4. *A unique equilibrium exists when $0 < |\mathbf{w}_1^T\mathbf{w}_2| < 1$.*

1. *When* $0 < \mathbf{w}_1^T\mathbf{w}_2 < 1$, $G_1^*(z_1(t)) = G_2^*(z_2(t))$ *and the support of* $\mathbf{Z}^*$ *is a segment on the line* $z_2 = vz_1$ *with* $\mathbf{z}(0) = \mathbf{0}$ *where*

$$v = \frac{\sqrt{[(u_2-u_1)\mathbf{w}_1^T\mathbf{w}_2]^2 + 4u_1u_2} - (u_2-u_1)\mathbf{w}_1^T\mathbf{w}_2}{2u_1} \geq 0. \tag{10}$$

2. *When* $-1 < \mathbf{w}_1^T\mathbf{w}_2 < 0$, $G_1^*(z_1(t)) + G_2^*(z_2(t)) = 1$ *and* $\frac{z_1(0)}{z_2(0)} = \frac{z_2(1)}{z_1(1)} = \mathbf{w}_1^T\mathbf{w}_2$.

When $0 < \mathbf{w}_1^T\mathbf{w}_2 < 1$, $Z_2^* = vZ_1^*$ and Condition (6) reduces to

$$\lambda^* + \max_{z_1}\left\{(u_1+u_2)[G_1^*(z_1)]^{n-1} - \lambda^* z_1^2 \Gamma\left((1,v)^T\right)\right\} = \frac{u_1+u_2}{n}.$$

Letting

$$\tilde{\mathbf{w}} = \mathbf{W}(\mathbf{W}^T\mathbf{W})^{-1}(1,v)^T = \frac{1 - v\mathbf{w}_1^T\mathbf{w}_2}{1-(\mathbf{w}_1^T\mathbf{w}_2)^2}\mathbf{w}_1 + \frac{v - \mathbf{w}_1^T\mathbf{w}_2}{1-(\mathbf{w}_1^T\mathbf{w}_2)^2}\mathbf{w}_2,$$

we have $\mathbf{X}^* = \mathbf{W}(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{Z}^* = Z_1^*\tilde{\mathbf{w}}$, $Z_1^* = \frac{\tilde{\mathbf{w}}^T\mathbf{X}^*}{\tilde{\mathbf{w}}^T\tilde{\mathbf{w}}}$, $G_1^*(z_1) = P_{F^*}\left(\frac{\tilde{\mathbf{w}}^T\mathbf{X}^*}{\tilde{\mathbf{w}}^T\tilde{\mathbf{w}}} \leq z_1\right)$, and $\Gamma((1,v)^T) = \tilde{\mathbf{w}}^T\tilde{\mathbf{w}}$. Thus, the equilibrium strategy F^* satisfies

$$\lambda^* + \max_{\mathbf{x}}\left\{(u_1+u_2)P_{F^*}(\tilde{\mathbf{w}}^T\mathbf{X}^* \leq \tilde{\mathbf{w}}^T\mathbf{x}) - \lambda^*\mathbf{x}^T\mathbf{x}\right\} = \frac{u_1+u_2}{n}.$$

That is, the problem is reduced to a single event one with a reward $u_1 + u_2$ and an attribute vector $\frac{\tilde{\mathbf{w}}}{\|\tilde{\mathbf{w}}\|_2}$ which lies inside the cone spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$ and improving a competitor's effort along $\tilde{\mathbf{w}}$ results in the highest expected total reward. If $u_1 = u_2$, then $v = 1$ and $\tilde{\mathbf{w}} = \frac{1}{1+\mathbf{w}_1^T\mathbf{w}_2}(\mathbf{w_1} + \mathbf{w_2})$ lies exactly in the middle of $\mathbf{w}_1$ and $\mathbf{w}_2$. As $\frac{u_2}{u_1}$ increases, $\tilde{\mathbf{w}}$ gets increasingly closer to $\mathbf{w}_2$. Figure 1 illustrates the supports and density functions of $\mathbf{Z}^*$ and $\mathbf{X}^*$.

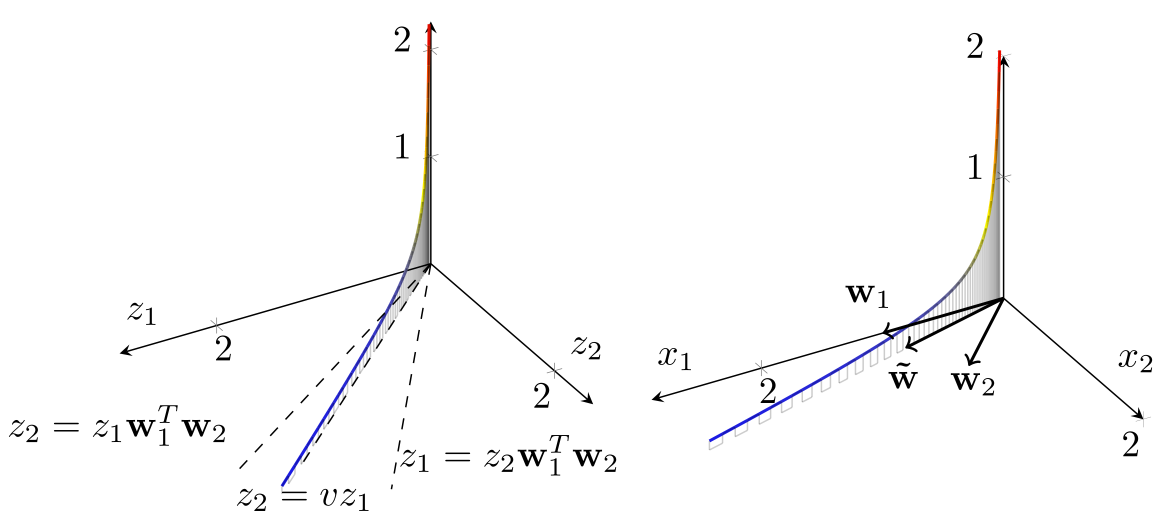


Figure 1 The supports and density functions of $\mathbf{Z}^*$ and $\mathbf{X}^*$ when $u_1 = 2u_2$ and $\mathbf{w}_1^T\mathbf{w}_2 = 0.7$

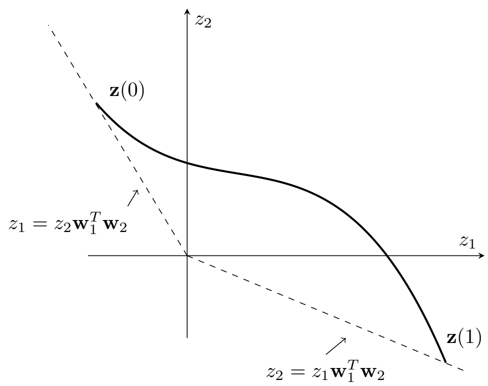


Figure 2 The support of $\mathbf{Z}^*$ when $z_2(t)$ is decreasing

When $-1 < \mathbf{w}_1^T\mathbf{w}_2 < 0$, we are unable to derive the explicit form of the support of $\mathbf{Z}^*$ as z_2 decreases in z_1 in a nonlinear fashion in general as illustrated in Figure 2. For instance, when $u_1 = u_2$ and $n = 2$, the solution to (EC.14)–(EC.15) must satisfy $z_1^2(t) + z_2^2(t) - 2z_1(t)z_2(t)\mathbf{w}_1^T\mathbf{w}_2 = 1 - \left(\mathbf{w}_1^T\mathbf{w}_2\right)^2$, i.e., the support is a segment of an ellipse and $G_j^*(z_j) = \frac{1}{2} + \frac{-\sin\left(2\theta_0+2\arcsin z_j\right)-2\mathbf{w}_1^T\mathbf{w}_2\left(\theta_0+\arccos z_j-\frac{\pi}{2}\right)}{2\sin 2\theta_0-4\mathbf{w}_1^T\mathbf{w}_2\theta_0}$ where $\theta_0 = \arcsin\left(\sqrt{\frac{1-\mathbf{w}_1^T\mathbf{w}_2}{2}}\right)$ and $G^*(\mathbf{z}) = [G_1^*(z_1) + G_2^*(z_2) - 1]^+$. Under the equilibrium strategy, each competitor is more likely to achieve a positive score for both events than sacrificing one event, as illustrated in Figure 3, where the density of $\mathbf{Z}^*$ is symmetric and the highest at the center of the support.

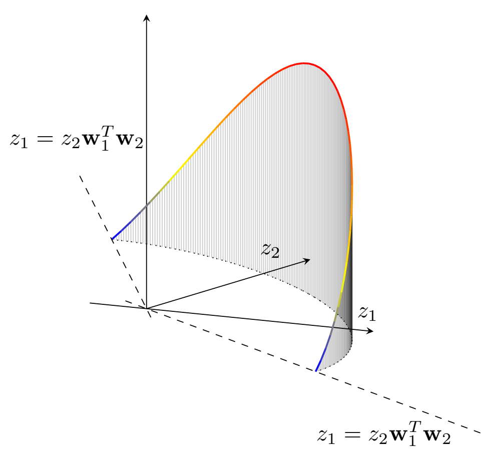


Figure 3 The density of $\mathbf{Z}^*$ on its support when $u_1 = u_2$, $n = 2$, and $\mathbf{w}_1^T\mathbf{w}_2 = -0.7$

However, we can provide some insights into its support and the distribution on the support as n becomes large in Proposition 5. Let $t_0 = \frac{\sqrt[n-2]{u_2}}{\sqrt[n-2]{u_1}+\sqrt[n-2]{u_2}}$.

PROPOSITION 5. *When, $-1 < \mathbf{w}_1^T\mathbf{w}_2 < 0$, for any given n and $\varepsilon \geq u_1 t_0^{n-1} \vee u_2(1-t_0)^{n-1}$, an ε-equilibrium strategy exists whose support lies on the boundary of the cone. Its density is unimodal on the support with $P(Z_1>0)=1-t_0$ and $P(Z_2>0)=t_0$, and reaches its peak at $\mathbf{z}=\mathbf{0}$. Furthermore, the strategy approaches G^* as $n\to\infty$.*

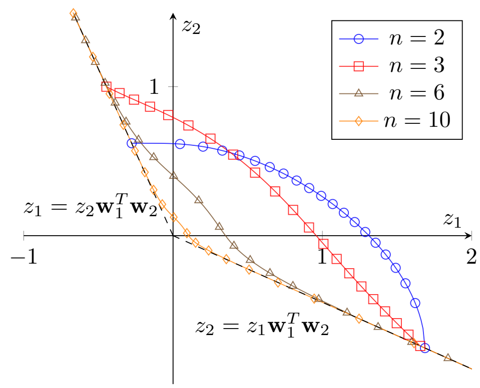


Figure 4 Support of $\mathbf{Z}^*$ for various n when $\mathbf{w_1}^T\mathbf{w}_2 = \frac{-1}{\sqrt{5}}$ and $u_1 = 2u_2$

As the number of competitors becomes large, the support of $\mathbf{Z}^*$ moves towards the boundary of the cone. Furthermore, $t_0 \to \frac{1}{2}$ as $n\to\infty$ and the equilibrium strategy assigns an equal probability to a positive z_1 and a positive z_2. However, the conditional tail probability $P(Z_j > z_j|Z_j>0)$ is lower than the unconditioned one under a single event scenario. With two events, a competitor has a positive probability of winning the other event and hence assigns a lower probability towards either tail.

4.1.3. When $\mathbf{w}_1^T\mathbf{w}_2 = \pm 1$ In these two cases, $\mathbf{w}_2 = \pm\mathbf{w}_1$ and condition (6) is equivalent to

$$\lambda^* + u_1\left[G_1^*(z_1)\right]^{n-1} + u_2\left[G_2^*(z_1\mathbf{w}_1^T\mathbf{w}_2)\right]^{n-1} - \lambda^* z_1^2 = \frac{u_1+u_2}{n} \tag{11}$$

on the support of Z_1^*.

THEOREM 2. *When $\mathbf{w}_1^T\mathbf{w}_2 = -1$ and $u_1 = u_2$, any feasible G^* is an equilibrium. Otherwise, a unique equilibrium exists when $\mathbf{w}_1^T\mathbf{w}_2 = \pm 1$.*

1. *When $\mathbf{w}_1^T\mathbf{w}_2 = 1$, $\lambda^* = \frac{u_1+u_2}{n}$ and $G_j^*(z_j) = \sqrt[n-1]{\frac{z_j^2}{n}}$, $z_j \in [0,\sqrt{n}]$, $j=1,2$.*
2. *When $\mathbf{w}_1^T\mathbf{w}_2 = -1$, $G_1^*(-z) + G_2^*(z) = 1$.*
   (a) *If $n=2$, $G_j^*(z_j) = \frac{z_j^2}{2}$, $z_j \in \left[0,\sqrt{2}\right]$ if $u_j > u_{3-j}$, $j=1,2$.*
   (b) *If $n>2$, $u_1\left[G_1^*(z_1)\right]^{n-1} + u_2[1-G_1^*(z_1)]^{n-1} = \lambda^*(z_1^2-1) + \frac{u_1+u_2}{n}$,*

$z_1 \in \left[-\sqrt{\frac{u_2+\lambda^*-\frac{u_1+u_2}{n}}{\lambda^*}}, \sqrt{\frac{u_1+\lambda^*-\frac{u_1+u_2}{n}}{\lambda^*}}\right]$, *where* $\lambda^* = \frac{u_1+u_2}{n} - u_1 t_0^{n-1} - u_2(1-t_0)^{n-1}$.

Recall that two equilibria may exist only when $\mathbf{w}_1^T\mathbf{w}_2 = -1$ by Theorem 1 and Proposition 1. With homogeneous competitors, Theorem 2 further establishes that multiple equilibria can only occur if $u_1 = u_2$ and $n = 2$, in which case (11) holds for any feasible G^* and $\lambda^* = 0$. Each event is won either by one competitor or both competitors (e.g., following a deterministic strategy as discussed in the paragraph following Proposition 1). In either case, each competitor gains half of the total rewards.

When $\mathbf{w}_1^T\mathbf{w}_2 = 1$, a unique equilibrium exists and the problem reduces to a single-event one with $\tilde{\mathbf{w}} = \mathbf{w}_1$, which has been considered by Alpern and Howard (2017) under a more general constraint. They show that $G_1^*(z_1) = \sqrt[n-1]{\frac{z_1^2}{n}}$ is a symmetric equilibrium, but only establish its uniqueness for $n = 2$. We are able to establish both the existence and uniqueness of the symmetric equilibrium for any given n. The variance of the score $Var\left(Z_1^*\right) = 1 - \frac{2\sqrt{n}}{n+1}$ and the size of the support $[0, \sqrt{n}]$ increase in n, implying that competitors are taking more risky actions to outperform more competitors, while the average score $E\left(Z_1^*\right) = \frac{2\sqrt{n}}{n+1}$ decreases in n. However, the distribution of the highest score $\left[G_1^*(z_1)\right]^n$ is stochastically increasing in n, suggesting that more contestants participating in a qualifying tournament of a major contest, e.g., an Olympic game, increases a country's winning chance in the game.

When $\mathbf{w}_1^T\mathbf{w}_2 = -1$, the problem cannot be reduced to a single-event one in general, because winning in one event means losing for sure in the other one and a competitor needs to strike a balance. However, when $n = 2$, the loser in one event becomes the winner in the other event and each competitor takes an event. Thus, the aim is to win the event with a higher reward and the problem reduces to one with a single event.

Unlike the case where $-1 < \mathbf{w}_1^T\mathbf{w}_2 < 0$, the problem when $\mathbf{w}_1^T\mathbf{w}_2 = -1$ can be reduced to a single-dimensional problem although it is not a single-event problem. This is because the cone itself in which the support of G^* lies becomes a straight line $z_1 = -z_2$ and the support is a segment on the line passing through the origin.

4.1.4. Summary of Symmetric Equilibria From the preceding analysis, we are ready to summarize the symmetric equilibria as a function of $\mathbf{D}$, $(\mathbf{w}_1, \mathbf{w}_2)$, and (u_1, u_2).

Theorem 3. *A symmetric equilibrium always exists.*

1. *If the two events are uncorrelated, i.e.,* $\mathbf{w}_1^T\mathbf{D}^{-1}\mathbf{w}_2 = 0$, *then,* $G_j^*(z_j) = \left(\frac{u_1+u_2}{nu_j}\frac{z_j^2}{\mathbf{w}_j^T\mathbf{D}^{-1}\mathbf{w}_j}\right)^{\frac{1}{n-1}}$ *with* $E\left[\frac{Z_j^2}{\mathbf{w}_j^T\mathbf{D}^{-1}\mathbf{w}_j}\right] = \frac{u_j}{u_1+u_2}$, $j = 1, 2$, *and any joint distribution* G^* *is an equilibrium if and only if it has the above marginals.*

2. *Otherwise, except when* $\mathbf{w}_1 = -\mathbf{w}_2$, $n = 2$, *and* $u_1 = u_2$, *in which case any feasible* G^* *is an equilibrium, a unique symmetric equilibrium exists.*

(a) *If* $\mathbf{w}_1^T\mathbf{D}^{-1}\mathbf{w}_2>0$, *then,* $Z_2^*=vZ_1^*$, *where* v *is the unique positive solution to the quadratic equation* $\mathbf{w}_1^T\mathbf{D}^{-1}\mathbf{w}_1u_1v^2+\mathbf{w}_1^T\mathbf{D}^{-1}\mathbf{w}_2(u_2-u_1)v-\mathbf{w}_2^T\mathbf{D}^{-1}\mathbf{w}_2u_2=0$ *and the problem is reduced to a single-event one with* $\frac{\tilde{\mathbf{w}}}{\|\tilde{\mathbf{w}}\|_2}$ *where*

$$\tilde{\mathbf{w}}=\begin{cases}(\mathbf{w}_2^T\mathbf{D}^{-1}\mathbf{w}_2-v\mathbf{w}_1^T\mathbf{D}^{-1}\mathbf{w}_2)\,\mathbf{w}_1+(-\mathbf{w}_1^T\mathbf{D}^{-1}\mathbf{w}_2+v\mathbf{w}_1^T\mathbf{D}^{-1}\mathbf{w}_1)\,\mathbf{w}_2, & \text{if } \mathbf{w}_1\neq\mathbf{w}_2,\\ \mathbf{w}_1, & \text{if } \mathbf{w}_1=\mathbf{w}_2.\end{cases}$$

(b) *If* $\mathbf{w}_1^T\mathbf{D}^{-1}\mathbf{w}_2<0$, *then,* G^* *has a decreasing support that approaches the boundary of the cone spanned by* $(1,\mathbf{w}_1^T\mathbf{D}^{-1}\mathbf{w}_2)$ *and* $(\mathbf{w}_1^T\mathbf{D}^{-1}\mathbf{w}_2,1)$ *as the number of competitors grows large.*

4.2. Two Types of Competitors

With heterogeneous competitors, a competitor may face both inter-type and intra-type competitions, and equilibrium strategies are asymmetric in general and difficult to obtain. Suppose that there are n_i type i competitors with $\mathbf{D}_i$ and homogeneous strategy $\mathbf{Z}^i\sim G^i$, $i=1,2$. We provide equilibrium solutions when they compete in two uncorrelated events in Section 4.2.1 and characterize necessary and sufficient conditions under which the problem can be reduced to a single event one in Section 4.2.2.

Recall that $\mathbf{w}_j^T\mathbf{D}_i^{-1}\mathbf{w}_j$ measures the competitiveness of type i competitors in event j and thus, $\frac{\mathbf{w}_1^T\mathbf{D}_i^{-1}\mathbf{w}_1}{\mathbf{w}_2^T\mathbf{D}_i^{-1}\mathbf{w}_2}$ measures type i competitors' relative competitiveness in event 1 over event 2. Without loss of generality, we assume that $\frac{\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}{\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2}\geq\frac{\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_1}{\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}$, i.e., compared with type 1 competitors, type 2 competitors have more advantage in event 1 over event 2. When $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent, condition (6) reduces to

$$\sum_{i=1}^{2}n_i\lambda_i^*+n_i\max_{\mathbf{z}}\left\{\sum_{j=1}^{2}u_jG_j^{i*}(z_j)^{n_i-1}G_j^{i'*}(z_j)^{n_{i'}}-\lambda_i^*\mathbf{z}^T\hat{\mathbf{D}}_i\mathbf{z}\right\}=u_1+u_2,\tag{12}$$

for $i,i'=1,2$, and $i\neq i'$.

4.2.1. When $\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_2=\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2=0$ In this case, the two events are uncorrelated for all competitors, although the events may share some or all attributes. When $\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_2=0$, a sufficient condition for $\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2=0$ to hold is that $\mathbf{w}_1$ and $\mathbf{w}_2$ are the eigenvectors of $\mathbf{D}_1^{-1}\mathbf{D}_2$, which is certainly true if $\mathbf{D}_2$ is proportional to $\mathbf{D}_1$. When $\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_2=\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2=0$, improving the performance at one event has no impact on that of the other for both types of competitors. Similar to the symmetric case with identical competitors, an infinite number of equilibria exists with the same marginal distributions as summarized in Proposition 6.

Proposition 6. *Suppose that* $\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_2=\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2=0$ *and* $\frac{\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}{\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2}\geq\frac{\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_1}{\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}$. *Any* (G^{1*},G^{2*}) *with the following marginal distributions is an equilibrium and type* 1 (2) *competitors always compete in event* 2 (1).

1. *If $n_i = n_{i'} = 1$, for $i, i' = 1, 2$, and $i \neq i'$,*

$$G_j^{i*}(z_j) = \frac{\frac{1}{2}\sum_{j=1}^{2} u_j \left(\frac{\mathbf{w}_j^T \mathbf{D}_2^{-1}\mathbf{w}_j}{\mathbf{w}_j^T \mathbf{D}_1^{-1}\mathbf{w}_j} \wedge \frac{\mathbf{w}_j^T \mathbf{D}_1^{-1}\mathbf{w}_j}{\mathbf{w}_j^T \mathbf{D}_2^{-1}\mathbf{w}_j}\right)}{u_j \mathbf{w}_j^T \mathbf{D}_i^{-1}\mathbf{w}_j} z_j^2 + \left(1 - \frac{\mathbf{w}_j^T \mathbf{D}_i^{-1}\mathbf{w}_j}{\mathbf{w}_j^T \mathbf{D}_{i'}^{-1}\mathbf{w}_j}\right)^+, \; j = 1, 2,$$

i.e., both competitors compete in both events, and competitor i's expected reward is

$$\sum_{j=1}^{2} u_j \left[\left(1 - \frac{\mathbf{w}_j^T \mathbf{D}_{i'}^{-1}\mathbf{w}_j}{\mathbf{w}_j^T \mathbf{D}_i^{-1}\mathbf{w}_j}\right)^+ + \frac{1}{2}\left(\frac{\mathbf{w}_j^T \mathbf{D}_2^{-1}\mathbf{w}_j}{\mathbf{w}_j^T \mathbf{D}_1^{-1}\mathbf{w}_j} \wedge \frac{\mathbf{w}_j^T \mathbf{D}_1^{-1}\mathbf{w}_j}{\mathbf{w}_j^T \mathbf{D}_2^{-1}\mathbf{w}_j}\right)\right].$$

2. *If $n_i > 1$, $i = 1, 2$, the marginal distributions are given in Proposition EC.1 in the Electronic Companion. Type 1 (2) competitors compete in both events if and only if $\frac{u_1 n_1}{u_2 n_2} > \frac{\mathbf{w}_2^T \mathbf{D}_2^{-1}\mathbf{w}_2}{\mathbf{w}_2^T \mathbf{D}_1^{-1}\mathbf{w}_2}$ $\left(\frac{u_2 n_2}{u_1 n_1} > \frac{\mathbf{w}_1^T \mathbf{D}_1^{-1}\mathbf{w}_1}{\mathbf{w}_1^T \mathbf{D}_2^{-1}\mathbf{w}_1}\right)$.*

3. *Otherwise, the marginal distributions are presented in Proposition EC.2 in the Electronic Companion.*
   - *The majority will always compete in both events.*
   - *The sole minority will compete in both events if and only if the relative reward from the event of her advantage to that of the other event, $\frac{u_2}{u_1}$ if $n_1 = 1$, is below a threshold. Furthermore, the threshold is decreasing in the size of the majority.*

If one type is a singleton, the sole competitor faces no intra-type competition and will win an event without any effort if the other type does not participate. Thus, if both types are singletons, the sole competitor in each type will compete in both events. Otherwise, if only one type is a singleton, the sole minority competitor may give up one event if its reward is relatively low and her advantage in the other event is high enough, while the majority will always participate in both events.

If there are at least two competitors in each type, the ratio $\frac{n_1}{n_2}$, which reflects the intensity of intra-type and inter-type competitions, and the ratio between the rewards $\frac{u_1}{u_2}$ jointly contribute to the equilibrium strategies. Each type of competitors will compete in both events if they are not significantly outnumbered and the event of their advantage does not provide a high enough reward.

4.2.2. When Can the Problem Be Reduced to a Single-Event One? It is obvious that when $\mathbf{w}_1 = \mathbf{w}_2$, in which case the two events differ only in their rewards or are identical, the problem is a single-event one with reward $u_1 + u_2$. When $\mathbf{w}_1 = -\mathbf{w}_2$, expect when $n_1 = n_2 = 1$ and $u_1 = u_2$, improving the performance along any single direction would result in a negative score in one of the events for all competitors. Thus, a competitor can win an event effortlessly by deviating from that direction and the problem cannot be reduced to a single-event one. When $\mathbf{w}_1 = -\mathbf{w}_2$, $n_1 = n_2 = 1$ and $u_1 = u_2$, it is easy to show that Theorem 3 still applies, i.e., any strategy is an equilibrium, including reducing the problem to any single-event one.

When $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent, i.e., $\mathbf{W}^T\mathbf{W}$ is non-singular, recall that

$$\hat{\mathbf{D}}_i = \left(\mathbf{W}^T\mathbf{D}_i^{-1}\mathbf{W}\right)^{-1} = \frac{1}{|\mathbf{W}^T\mathbf{D}_i^{-1}\mathbf{W}|}\begin{pmatrix} \mathbf{w}_2^T\mathbf{D}_i^{-1}\mathbf{w}_2 & -\mathbf{w}_1^T\mathbf{D}_i^{-1}\mathbf{w}_2 \\ -\mathbf{w}_1^T\mathbf{D}_i^{-1}\mathbf{w}_2 & \mathbf{w}_1^T\mathbf{D}_i^{-1}\mathbf{w}_1 \end{pmatrix}$$
$$\triangleq \frac{1}{|\mathbf{W}^T\mathbf{D}_i^{-1}\mathbf{W}|}\begin{pmatrix} d_{22}^i & d_{12}^i \\ d_{12}^i & d_{11}^i \end{pmatrix}$$

as defined in (6). Proposition 7 provides a necessary and sufficient condition under which the problem can be reduced to a single-event one. Similar to Theorem 3, let

$$\tilde{\mathbf{w}} = \left(d_{22}^1 - v d_{12}^1\right)\mathbf{w}_1 + \left(v d_{11}^1 - d_{12}^1\right)\mathbf{w}_2,$$

and v is the unique positive solution to $d_{11}^1 u_1 v^2 + d_{12}^1(u_2 - u_1)v - d_{22}^1 u_2 = 0$.

Proposition 7. *A problem can be reduced to a single-event one if and only if $d_{12}^i \geq 0$, $i = 1,2$, and*

$$\left(\frac{d_{11}^2}{d_{22}^2} - \frac{d_{11}^1}{d_{22}^1}\right)\left(\frac{d_{22}^1}{d_{11}^1} - \frac{d_{22}^2}{d_{11}^2}\right) = \frac{(u_2-u_1)^2}{u_1 u_2}\left(\frac{d_{12}^1}{d_{22}^1} - \frac{d_{12}^2}{d_{22}^2}\right)\left(\frac{d_{12}^1}{d_{11}^1} - \frac{d_{12}^2}{d_{11}^2}\right). \tag{13}$$

The single event has the attribute vector $\frac{\tilde{\mathbf{w}}}{\|\tilde{\mathbf{w}}\|_2}$ and reward $u_1 + u_2$, and an equilibrium is given in Proposition EC.3 in the Electronic Companion.

Proposition 7 further excludes the possibility of any negatively correlated events to be reduced to a single-event one, because a competitor is better off balancing two negatively correlated events than aiming to improve the performance in both events simultaneously. When the two events are either positively correlated or uncorrelated, i.e., $d_{12}^i \geq 0$, $i = 1,2$, condition (13) ensures a perfect balance between the rewards from and efforts required in both events for both types of competitors. Below are some examples.

1. $u_1 = u_2$ and $d_{11}^i = d_{22}^i$ for $i = 1,2$, i.e., the rewards from and the effort required in the two events are the same for all competitors, in which case, $v = 1$ and $\tilde{\mathbf{w}}$ is proportional to $\mathbf{w}_1 + \mathbf{w}_2$.

2. $d_{12}^1 = d_{12}^2 = 0$, i.e., the events are uncorrelated for both types of competitors, in which case condition (13) implies $\frac{d_{11}^2}{d_{22}^2} = \frac{d_{11}^1}{d_{22}^1}$, i.e., the relative advantage of event 1 over event 2 is the same for both types of competitors, and $\tilde{\mathbf{w}}$ is proportional to $\sqrt{\frac{u_1}{d_{11}^1}}\mathbf{w}_1 + \sqrt{\frac{u_2}{d_{22}^1}}\mathbf{w}_2$.

3. $\frac{d_{11}^1}{d_{22}^1} = \frac{d_{11}^2}{d_{22}^2}$, i.e. the relative advantage of event 1 over event 2 is the same for both types of competitors, in which case condition (13) implies that either $u_1 = u_2$ or $\frac{d_{12}^1}{d_{jj}^1} = \frac{d_{12}^2}{d_{jj}^2}$, $j = 1,2$, i.e., the relative ability of improving the performance in both events simultaneously is the same for both types of competitors.

Alpern and Howard (2017) studied a single-event problem with two competitors and a more generalized constraint, and the case where $n_1 = n_2 = 1$ of Proposition 7 is a special case of their Corollary 5. However, the equilibrium solutions in the Electronic Companion for the case where either $n_1 > 1$ and/or $n_2 > 1$ may shed some light on the single-event problems in their paper for more than two competitors and different budget constraints.

4.3. Two Heterogeneous Competitors

This is the case with $n_1 = n_2 = 1$ in Section 4.2 for which we have a complete characterization of the equilibrium structure when $\mathbf{w}_1^T \mathbf{D}_1^{-1} \mathbf{w}_2 = \mathbf{w}_1^T \mathbf{D}_2^{-1} \mathbf{w}_2 = 0$ in Proposition 6 and when the problem can be reduced to a single-event one in Proposition 7. In this section, we develop an algorithm to calculate an equilibrium under more general conditions and conduct a numerical study to reveal some properties of the equilibria.

4.3.1. Algorithm for Finding an Equilibrium Suppose that the score Z_j^{i*} takes discrete values in $\mathcal{S} = \{0, \pm 1, \pm 2, \dots, \pm S\}$, $i = 1, 2$ and $j = 1, 2$. Since the optimization problem in (6) can be decomposed into two convex optimization problems for $i = 1, 2$, we will solve each of them through the ellipsoid method which requires the creation of a separation and subgradient oracle as follows. The method starts with an ellipsoid that includes all the feasible solutions and the separation oracle tests the feasibility of the center of the ellipsoid. If it is (not) feasible, the subgradient (separation) oracle generates a hyperplane that separates some suboptimal (infeasible) points from the rest of the ellipsoid. The process continues until the remaining ellipsoid is small enough.

Separation Oracle When $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent or $\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_2 = 0$, the left hand side of constraint (8) is a linear function and easy to solve. Otherwise, for any given $(G_1^i(\cdot), G_2^i(\cdot))$, constraint (8) is equivalent to the following linear constraint following a similar argument as in the proof of Proposition 2,

$$\sum_{j=1}^{2} \sum_{z_j \in \mathcal{S}} \varphi_j^i(z_j) G_j^i(z_j) \leq 1, \tag{14}$$

where

$$\varphi_1^i(z_1) = \begin{cases} \Gamma_i\left(z_1, \max\left\{z_2 : G_2^i(z_2) \leq G_1^i(z_1)\right\}\right) - \Gamma_i\left(z_1 + 1, \max\left\{z_2 : G_2^i(z_2) \leq G_1^i(z_1)\right\}\right), & z_1 < S, \\ \frac{\Gamma_i(S,S)}{2}, & z_1 = S, \end{cases}$$

$$\varphi_2^i(z_2) = \begin{cases} \Gamma_i\left(\min\left\{z_1 : G_1^i(z_1) > G_2^i(z_2)\right\}, z_2\right) - \Gamma_i\left(\min\left\{z_1 : G_1^i(z_1) > G_2^i(z_2)\right\}, z_2 + 1\right), & z_2 < S, \\ \frac{\Gamma_i(S,S)}{2}, & z_2 = S, \end{cases}$$

when $\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_2 > 0$ and

$$\hat{\varphi}_{1,z_1}^i = \begin{cases} \Gamma_i\left(z_1, \min\left\{z_2 : G_1^i(z_1) + G_2^i(z_2) \geq 1\right\}\right) - \Gamma_i\left(z_1 + 1, \min\left\{z_2 : G_1^i(z_1) + G_2^i(z_2) \geq 1\right\}\right), & z_1 < S, \\ \Gamma_i(S, -S), & z_1 = S, \end{cases}$$

$$\hat{\varphi}_{2,z_2}^i = \begin{cases} \Gamma_i\left(\min\left\{z_1 : G_1^i(z_1) + G_2^i(z_2) \geq 1\right\}, z_2\right) - \Gamma_i\left(\min\left\{z_1 : G_1^i(z_1) + G_2^i(z_2) \geq 1\right\}, z_2 + 1\right), & z_2 < S, \\ -\sum_{z_2'=-S}^{S-1} \varphi_{2,z_2'}^i, & z_2 = S, \end{cases}$$

when $\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_2 < 0$. Thus, feasibility of a solution can be easily checked. If a solution is not feasible, one can cut the half space defined by the linear constraint from the ellipsoid.

Subgradient Oracle Suppose that $(\hat{\lambda}_{i'}, \hat{G}_1^i(\cdot), \hat{G}_2^i(\cdot))$ is a feasible solution and $\mathbf{z}^{i*}$ is an optimal solution to the maximization problem in (6). Then, any point in the half space defined by

$$\lambda_{i'} + \sum_{j=1}^{2} u_j \frac{G_j^i(z_j^i - 1) + G_j^i(z_j^i)}{2} - \lambda_{i'}\Gamma_{i'}(\mathbf{z}^{i*}) > \hat{\lambda}_{i'} + \sum_{j=1}^{2} u_j \frac{\hat{G}_j^i(z_j^i - 1) + \hat{G}_j^i(z_j^i)}{2} - \hat{\lambda}_{i'}\Gamma_{i'}(\mathbf{z}^{i*})$$

cannot be an optimal solution.

Next, we conduct a numerical study using the above algorithm to examine the equilibrium supports and distribution functions for various combinations of the parameters. Since the problem is equivalent to one with $\tilde{\mathbf{w}}_j = \frac{\sqrt{\mathbf{D}_1}^{-1}\mathbf{w}_j}{\left\|\sqrt{\mathbf{D}_1}^{-1}\mathbf{w}_j\right\|_2}$, $\tilde{\mathbf{D}}_1 = \mathbf{I}$, $\tilde{\mathbf{D}}_2 = \sqrt{\mathbf{D}_1}^{-1}\mathbf{D}_2\sqrt{\mathbf{D}_1}^{-1}$, we assume $\mathbf{D}_1 = \mathbf{I}$ without loss of generality and vary $(u_1, u_2, \mathbf{w}_1, \mathbf{w}_2, \mathbf{D}_2)$.

4.3.2. Numerical Results

When $\mathbf{D}_2 = d\mathbf{I}$, $d > 0$, a Scalar Matrix

In this case, competitor 2 is weaker (stronger) than competitor 1 in both events if $d > 1$ ($d < 1$) and the two events are correlated for both competitors if $\mathbf{w}_1^T\mathbf{w}_2 \neq 0$ and uncorrelated otherwise, in which case any distribution with the marginals specified in Proposition 6 is an equilibrium.

We plot the equilibrium probability mass functions on their supports by the intensity of the colors for $d = 2$, $\mathbf{w}_1^T\mathbf{w}_2 = \pm\frac{2}{3}, \pm\frac{1}{3}$ and $\frac{u_2}{u_1} = 1, 10$ in Table 1. When $\mathbf{w}_1^T\mathbf{w}_2 > 0$, the problems can be reduced to a single-event one and thus, the supports of the equilibria are increasing sets on straight lines. Competitor 2 as the weaker one has a higher probability of losing both events, although both competitors gain a higher score in expectation as $\mathbf{w}_1^T\mathbf{w}_2$ increases. When $\mathbf{w}_1^T\mathbf{w}_2 < 0$, the supports are decreasing sets and do not include the origin in all the examples, implying that a competitor can still obtain relatively low positive scores in both events. The supports of competitor 2, the weaker one, always lie below those of competitor 1, implying lower scores in both events. However, negative correlation of the events give the weaker competitor a better chance of winning one event.

Note also that, when $u_1 = u_2$, the supports are symmetric. That is, competitors aim to achieve the same score in both events when $\mathbf{w}_1^T\mathbf{w}_2 > 0$, while they balance the scores from the two events symmetrically when $\mathbf{w}_1^T\mathbf{w}_2 < 0$. As u_2 increases, both competitors shift their effort toward event 2 as expected when $\mathbf{w}_1^T\mathbf{w}_2 > 0$. When $\mathbf{w}_1^T\mathbf{w}_2 < 0$, the stronger competitor focuses more on event 2 while the weaker one has a lower chance of winning event 2.

Figure 5 plots competitor 1's expected share of the total reward as a function of $\mathbf{w}_1^T\mathbf{w}_2$ for $d = 1.5$, 3, 4.5, and $\frac{u_2}{u_1} = 1$, 10. For a given d, competitor 1's expected share is increasing when $\mathbf{w}_1^T\mathbf{w}_2 \leq 0$, i.e., the weaker competitor benefits from a stronger negative correlation as it is less likely for the stronger competitor to win both events, especially when the rewards from the two events are close. When $\mathbf{w}_1^T\mathbf{w}_2 > 0$, the problem is reduced to a single-event one and a competitor's probability of winning is purely determined by her advantage over the other, d in this case, regardless of the level of correlation. Hence, the competitors' shares remain constant.

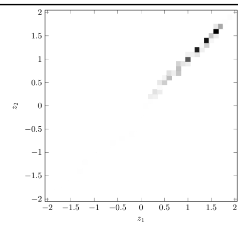
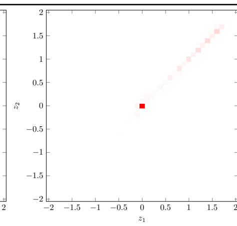
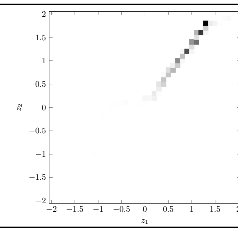
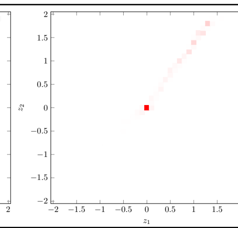
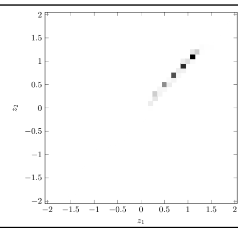
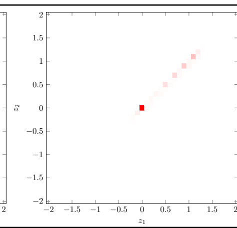
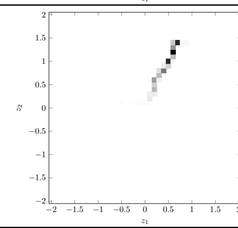
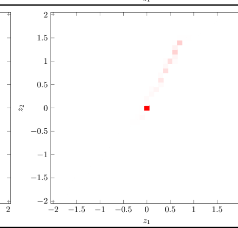
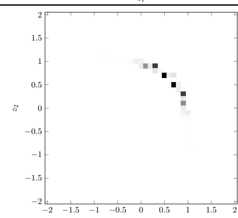
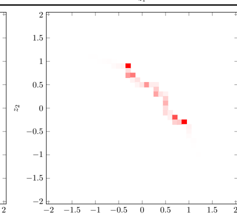
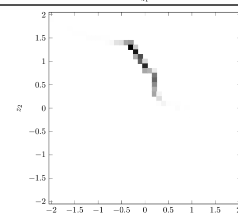
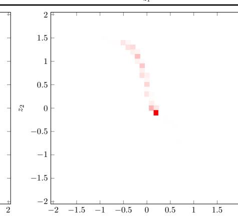
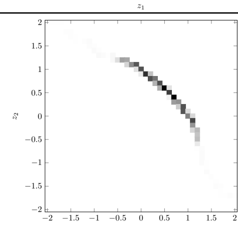
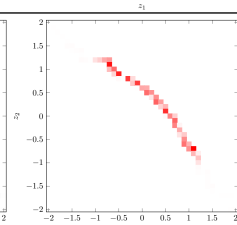
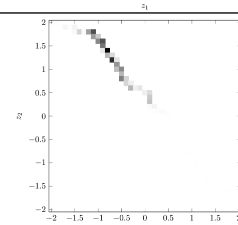
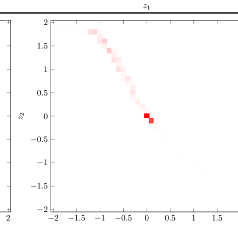
	$u_2 = u_1$		$u_2 = 10u_1$	
i	1	2	1	2
$\mathbf{w}_1^T\mathbf{w}_2 = \frac{2}{3}$				
$\mathbf{w}_1^T\mathbf{w}_2 = \frac{1}{3}$				
$\mathbf{w}_1^T\mathbf{w}_2 = -\frac{1}{3}$				
$\mathbf{w}_1^T\mathbf{w}_2 = -\frac{2}{3}$				

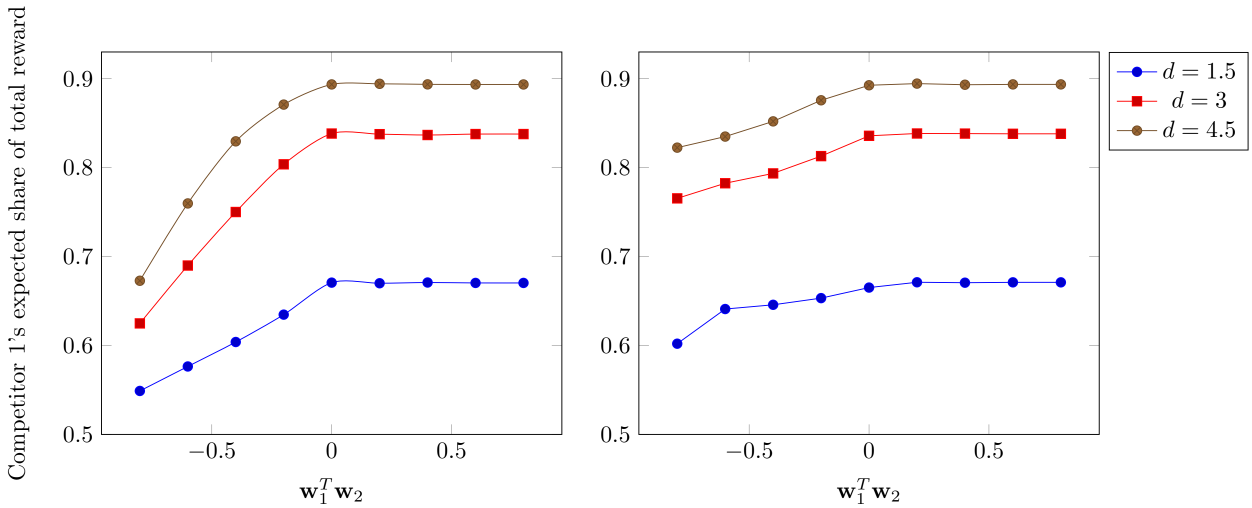
Table 1 Equilibrium distributions when $\mathbf{D}_2 = 2\mathbf{I}$

For general $\mathbf{D}_2$

To focus on the impact of the correlations, we examine cases with $\mathbf{w}_j^T\mathbf{D}_2^{-1}\mathbf{w}_j = 1$ for $j = 1, 2$, i.e., the competitors are equally strong in both events. In this case, $\mathbf{W}^T\mathbf{W} = \begin{pmatrix} 1 & \mathbf{w}_1^T\mathbf{w}_2 \\ \mathbf{w}_1^T\mathbf{w}_2 & 1 \end{pmatrix}$ and $\mathbf{W}^T\mathbf{D}_2^{-1}\mathbf{W} = \begin{pmatrix} 1 & \mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2 \\ \mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2 & 1 \end{pmatrix}$.

We fix $\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2 = 0.8$, and demonstrate the equilibrium probability mass functions by the intensity of the colors on their supports for $\mathbf{w}_1^T\mathbf{w}_2 = \pm 0.6, \pm 0.2$ and $\frac{u_2}{u_1} = 1, 10$ in Figure 6. As one can see, when $\mathbf{w}_1^T\mathbf{w}_2 > 0$, condition (13) is met and the problem can be reduced to a single-event one if and only if $u_1 = u_2$, in which case, the supports of the competitors' equilibrium distributions are different. The supports are not even straight lines when $u_1 \neq u_2$. When $\mathbf{w}_1^T\mathbf{w}_2 < 0$ and $u_1 \neq u_2$, competitor 1 focuses on the event with a higher reward while competitor 2 does not give up any event.

Figure 7 plots competitor 1's expected share of the total reward as a function of $\mathbf{w}_1^T\mathbf{w}_2$ for $\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2 = -0.5$, 0, 0.5, and $\frac{u_2}{u_1} = 1$, 10. Competitor 1's expected share strictly increases in $\mathbf{w}_1^T\mathbf{w}_2$



(a) $u_2 = u_1$

(b) $u_2 = 10u_1$

Figure 5 Competitor 1's expected shares of the total reward when $\mathbf{D}_2 = d\mathbf{I}$

and is equal to 0.5 when $\mathbf{w}_1^T\mathbf{w}_2 = \mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2$, implying that a higher positive correlation of the events always leads to a higher reward. The total reward is more evenly distributed between the two competitors when $\frac{u_1}{u_2}$ or $\frac{u_2}{u_1}$ becomes large.

5. More Than Two Events

We first establish in Section 5.1 that, if a set of events are linearly dependent, then the problem can be approximated by one with linearly independent events. We then provide equilibrium solutions under certain conditions when all the events are linearly independent in Section 5.2.

5.1. Linearly Dependent Events

If $\mathbf{w}_1, \ldots, \mathbf{w}_J$ are linearly dependent, consider a problem with the attribute vectors $\tilde{\mathbf{w}}_j^{(k)} = \begin{pmatrix} \mathbf{w}_j \\ \frac{1}{k}\mathbf{e}_j \end{pmatrix}$ where $k > 0$ and $\mathbf{e}_j \in \mathbb{R}^J$ is the vector with 1 in the jth coordinate and 0 elsewhere, and $\tilde{\mathbf{D}}_i^{(k)} = \begin{pmatrix} \mathbf{D}_i & 0 \\ 0 & k\mathbf{I}_J \end{pmatrix}$ where $\mathbf{I}_J$ is a J dimensional identity matrix. That is, we add a unique dummy attribute to each event such that any effort to improve it will have a negligible effect on winning the event when k is large enough, and $\tilde{\mathbf{w}}_1^{(k)}, \ldots, \tilde{\mathbf{w}}_J^{(k)}$ are linearly independent. Proposition 8 establishes that a solution to the original problem is a limit of solutions to the modified problems, denoted as $(G^{k1*}, \ldots, G^{kn})$.

Proposition 8. *A subsequence of* $\left\{(G^{k1}, \ldots, G^{kn}) : k = 1, 2, \ldots\right\}$ *converges to an equilibrium of the original problem.*

As k grows large, competitors will eventually stop investing in enhancing the dummy attributes as such an effort only consumes resources without improving the probabilities of winning the events.

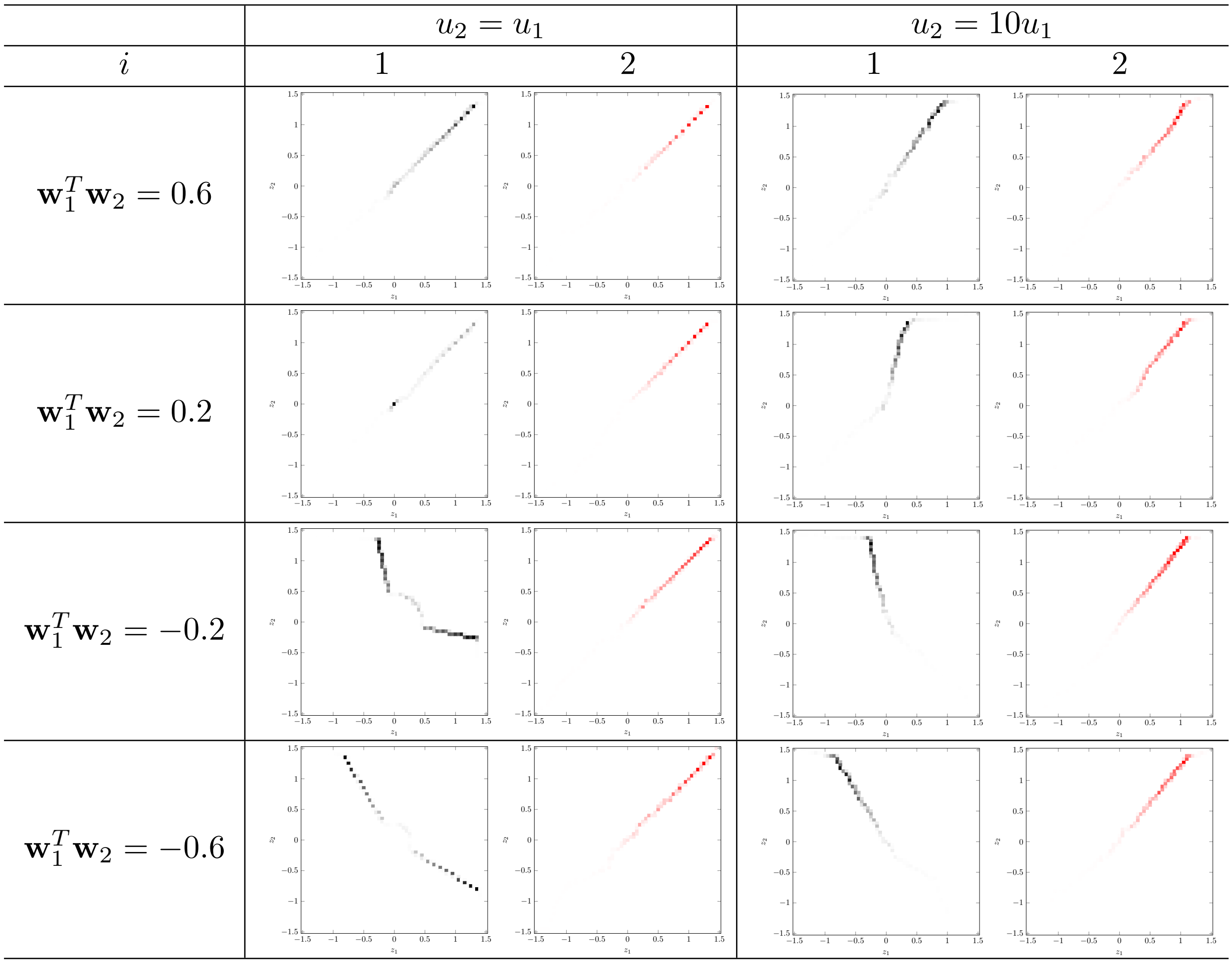


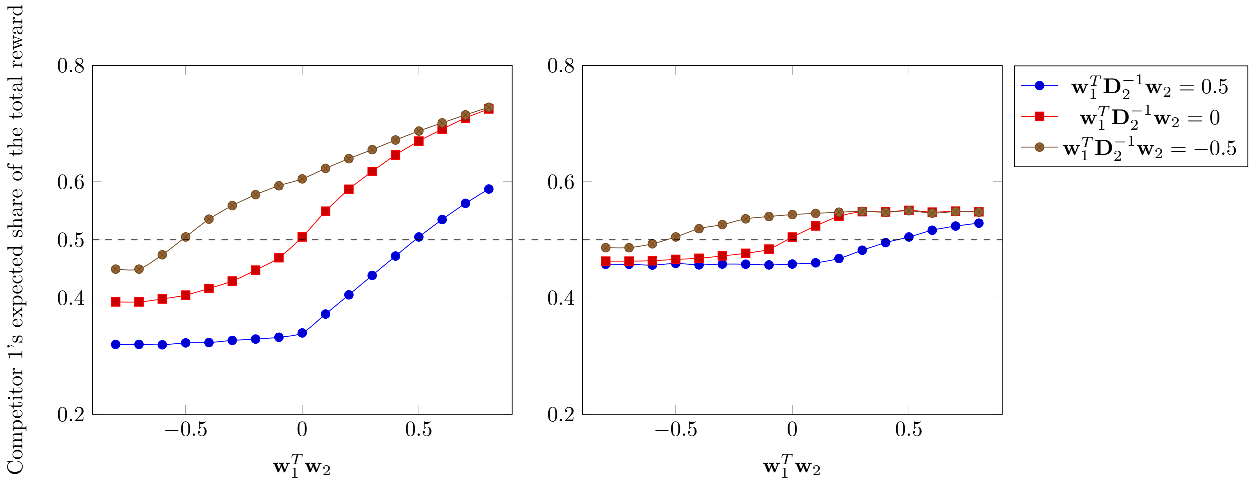
Figure 6 Equilibrium distributions when $\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2 = 0.8$

In the next section, we provide equilibrium solutions to problems with linearly independent events and homogeneous competitors.

5.2. Linearly Independent Events and Homogeneous Competitors

We show analytically in Proposition 9 that the problem of finding symmetric equilibria can be reduced to a single-event one if all the events are pairwise positively correlated or uncorrelated because the support of an equilibrium distribution is still a line segment given in the proof in the Electronic Companion. Furthermore, we identify an ε-equilibrium when all the events are pairwise negatively correlated or uncorrelated, in which case the support is a union of J line segments $\left\{t\sqrt{\mathbf{D}}^{-1}\mathbf{w}_j | t \geq 0\right\}$, $j = 1, 2, \ldots, J$. This ε-equilibrium implies that focusing on a randomly chosen event, regardless of its reward, is a good strategy when all the events are negatively correlated and there is a sufficient number of competitors.

Proposition 9. *Suppose that* $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_J$ *are linearly independent.*



(a) $u_2 = u_1$

(b) $u_2 = 10u_1$

Figure 7 Competitor 1's expected share of the total reward for $\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2 = -0.5,\, 0,\, 0.5$

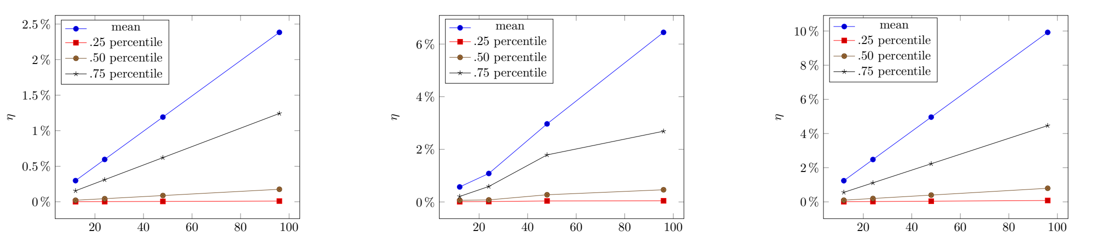
1. *If* $\mathbf{w}_j^T\mathbf{D}^{-1}\mathbf{w}_{j'} \geq 0$ *for all* j *and* j'*, the problem can be reduced to a single-event one.*
2. *If* $\mathbf{w}_j^T\mathbf{D}^{-1}\mathbf{w}_{j'} \leq 0$ *for all* j *and* j'*, for any given* $\varepsilon \geq \left(\frac{J-1}{J}\right)^n \sum_{j=1}^{J} u_j$*, an* ε*-equilibrium exists which assigns an equal probability mass,* $\frac{1}{J}$*, on a segment of the line* $\left\{t\sqrt{\mathbf{D}}^{-1}\mathbf{w}_j | t \geq 0\right\}$*,* $j = 1, 2, \ldots, J$*.*

Proposition 9 suggests the following heuristic strategies when there are multiple events and competitors are homogeneous.

1. If all the events are pairwise positively correlated or uncorrelated, identifying the equivalent single event involves solving a complicated system of multivariate quadratic equations. Thus, we suggest the following procedure to find a solution efficiently. We first find the equivalent single event for two events with the highest correlation using Proposition 4 and repeat the process until there is a single event.
2. If all the events are negatively correlated or uncorrelated, we follow the ε-equilibrium strategy $G(\mathbf{z})$ where $\mathbf{Z} = \mathbf{W}^T\mathbf{X}$ is such that $P(\mathbf{X} \in \{t\sqrt{\mathbf{D}}^{-1}\mathbf{w}_j | t \in [0, z_j]\}) = 1 \wedge \left[\left(\frac{J-1}{J}\right)^{n-1} + \frac{\hat{\lambda}}{u_j} z_j^2\right]^{\frac{1}{n-1}} - \frac{J-1}{J}$ where $\hat{\lambda} = \frac{J^n + (n-1)(J-1)^n - nJ(J-1)^{n-1}}{nJ^n} \sum_{j=1}^{J} u_j$ for any $j \in \mathcal{J}$ and $z_j \geq 0$ provided in the proof of Proposition 9.
3. For general events, we start with two events with the highest positive correlation and replace them with the equivalent single event and continue the process until all the events are pairwise negatively correlated or uncorrelated for which we adopt the strategy in 2.

As Proposition 9 provides a theoretical guarantee for the performance of the heuristic under scenario 2, we only need to evaluate the effectiveness of our heuristic strategies under scenarios 1 and 3. Since the expected reward for each competitor is $\frac{1}{n}\sum_{j\in\mathcal{J}} u_j$ under all the heuristic strategies, we conduct a numerical study to compare $\frac{1}{n}\sum_{j\in\mathcal{J}} u_j$ against the highest expected reward if a competitor deviates from the heuristic strategies, denoted as R^* and exactly the left hand side of (6) when all but one competitor adopt a heuristic strategy. We calculate the potential percentage gain of deviating from a heuristic strategy $\eta = \frac{R^*}{\frac{1}{n}\sum_{j\in\mathcal{J}} u_j} - 1$. We consider events with $m = 2$, i.e., there is a total of two attributes, and uniformly generate each attribute vector $\mathbf{w}_j$ from the set of non-negative unit vectors and its reward u_j from a folded normal distribution.

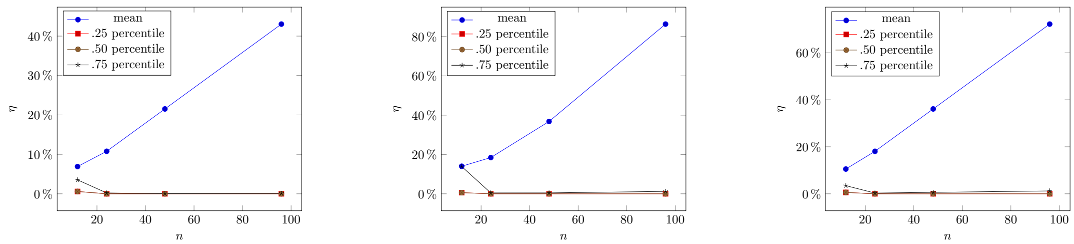
Under scenario 1, we consider problems with $J = 6$. Figure 8(a) presents the average and percentiles of ε over 1000 incidents for $n = 12, 24, 48, 96$. Evidently, our heuristic performs extremely well although η increases as n becomes large, in which case the potential gain by deviating from the heuristic is higher as the reward $\frac{1}{n}\sum_{j\in\mathcal{J}} u_j$ decreases in n much faster than R^*. Furthermore, merging events according to their correlations is significantly more effective than in a random order as shown in Figure 8(b) or according to their rewards from the highest to the lowest as shown in Figure 8(c). This result indicates that correlations are more important than rewards in identifying a good strategy. To see this, suppose that there is cluster consisting of a large number of highly correlated events, each with a low reward while their aggregated reward is extremely high, and another event with a medium reward and low correlation with this cluster of events. Merging the events according to their rewards would end up with a single event closer to this medium-reward event, while ordering according to correlations leads to a single event closer to the high-reward cluster.



(a) The Heuristic (b) Random Order (c) Order by Rewards

Figure 8 Average and Percentiles of Potential Percentage Gain with Positively Correlated Events

For scenario 3, we consider problems with $J = 3$. Figure 9 presents the average and percentiles of η over 1000 incidents for $n = 12, 24, 48, 96$. Intuitively, if we can successfully group events into negatively correlated clusters such that events in each cluster are "highly" positively correlated, then our heuristic strategy should perform well, especially when n is large, as implied by Proposition 9. Otherwise, a significant amount of information might be lost when merging weakly positively correlated events, which is the reason for the high average of η in Figure 9.



(a) The Heuristic (b) Random Order (c) Order by Reward

Figure 9 Average and Percentiles of Potential Percentage Gain with Events of General Correlations

Consider the example where $(u_1,u_2,u_3) = (2,0.5,1)$ and $(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3) = \begin{pmatrix} 1 & 0.3 & 0.3 \\ 0 & \sqrt{0.91} & -\sqrt{0.91} \end{pmatrix}$, as illustrated in Figure 10. In this case, $\mathbf{w}_1$ is weakly positively correlated with $\mathbf{w}_2$ and $\mathbf{w}_3$, while

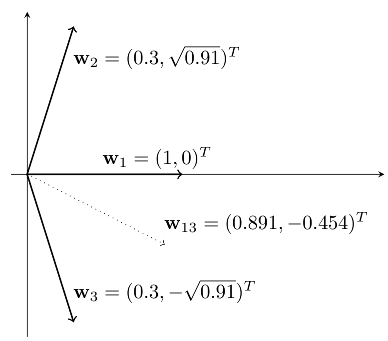


Figure 10 An example with three events

$\mathbf{w}_2$ and $\mathbf{w}_3$ are negatively correlated. Clustering as described above is not possible. Suppose that we merge $\mathbf{w}_1$ and $\mathbf{w}_3$ first to $\mathbf{w}_{13} = (0.891, -0.454)$ with reward $u_{13} = 3$. Then, $\mathbf{w}_{13}$ is negatively correlated with $\mathbf{w}_2$ and the positive correlation between $\mathbf{w}_1$ and $\mathbf{w}_2$ is lost, which results in $\eta = \frac{R^*}{\frac{1}{n}(u_1+u_2+u_3)} - 1 \approx 10.68$. In general, our heuristic fails when the events are more evenly spread out or weakly correlated. In fact, it is not clear what an equilibrium solution looks like and how to find one, which reflects many difficult decisions we face in real life.

6. Conclusions

Organizations and individuals often engage in multiple competitions or events simultaneously and a competitor wins a competition by outperforming her opponents, regardless of the winning margin. Since winning a competition comes with a reward, competitors will exert effort by consuming resources to improve their winning chances. Since resources are often limited and competitions may be correlated due to shared attributes, e.g., improving an attribute may have different or even conflicting effects on different competitions, competitors need to allocate their limited resources wisely. The resource allocation decision is further complicated as the outcome of a competition can be affected by external random factors and it is not clear how the winning probability functions of the competitions should depend on a competitor's abilities in the attributes. In this study, we study competitors' equilibrium distributions of their states in the attributes and their supports. We first describe mathematically the competitor-specific correlation between any pair of competitions and model the problem as a zero-sum game. By exploring some properties of the problem, we are able to reduce the dimension of the decisions and convert the game into a single non-convex optimization problem.

For the case with two events, we first analyze competitors' symmetric equilibrium decision with homogeneous competitors. We show that the problem can be reduced to a single-event one if the two events are positively correlated. If the two events are negatively correlated, we establish the existence and uniqueness of the equilibrium solution and, as the number of competitors becomes large, that each competitor will randomly choose an event to focus on. We then analyze the equilibrium decisions when there are two types of competitors, in which case, a competitor may face both intra-type and inter-type competitions and the problem becomes very challenging. We are able to derive the equilibrium solutions if the two events are uncorrelated for both types of competitors, and provide a necessary and sufficient condition under which the problem can be reduced to a single-event one. Numerical studies on the case with two heterogeneous competitors, i.e., there is a single competitor in each type, further shed some light on problems with heterogeneous competitors.

When there are more than two events and competitors are homogeneous, we show that the problem can be reduced to a single-event one if the events are pair-wise positively correlated. When the events are pare-wisely negatively correlated, we are able to construct an approximate equilibrium solution that works well when the number of competitors is large enough. These results suggest efficient heuristic strategies for general problems with general events and homogeneous competitors.

Future research can allow for more general objective functions and resource constraints. We believe that such a problem can be approached by exploiting structural properties of the underlying transportation problem. For example, if we replace constraint (2) by $E\left[\Phi\left(\mathbf{X}^T\mathbf{D}_i\mathbf{X}\right)\right] \leq 1$ for some differentiable increasing function Φ, the support of the equilibrium solution may be on the same line we identified and the problems can still be reduced to a single-event one with two positively correlated events and homogeneous competitors.

References

Ahmadinejad A, Dehghani S, Hajiaghayi M, Lucier B, Mahini H, Seddighin S (2019) From duels to battlefields: Computing equilibria of blotto and other games. *Mathematics of Operations Research* 44(4):1304–1325.

Alpern S, Howard J (2017) Winner-take-all games: the strategic optimisation of rank. *Operations Research* 65(5):1165–1176.

Anderson E (2012) Ranking games and gambling: When to quit when you're ahead. *Operations research* 60(5):1229–1244.

Bell RM, Thomas M (1980) Competitive optimality of logarithmic investment. *Mathematics of Operations Research* 5(2).

Boix-Adserà E, Edelman BL, Jayanti S (2020) The multiplayer colonel blotto game. *arXiv preprint arXiv:2002.05240* .

Borel E (1953) The theory of play and integral equations with skew symmetric kernels. *Econometrica: journal of the Econometric Society* 97–100.

Corchón LC, Serena M (2018) Contest theory. *Handbook of Game Theory and Industrial Organization, Volume II* (Edward Elgar Publishing).

Dziubiński M (2013) Non-symmetric discrete general lotto games. *International Journal of Game Theory* 42(4):801–833.

Hart S (2008) Discrete colonel blotto and general lotto games. *International Journal of Game Theory* 36(3-4):441–460.

Hart S (2016) Allocation games with caps: from captain lotto to all-pay auctions. *International Journal of Game Theory* 45(1-2):37–61.

Kovenock D, Roberson B (2010) Conflicts with multiple battlefields .

Kovenock D, Roberson B (2020) Generalizations of the general lotto and colonel blotto games. *Economic Theory* 1–36.

Macdonell ST, Mastronardi N (2015) Waging simple wars: a complete characterization of two-battlefield blotto equilibria. *Economic Theory* 58(1):183–216.

Mitroi FC, Niculescu CP (2011) An extension of young's inequality. *Abstract and Applied Analysis*, volume 2011 (Hindawi).

Myerson RB (1993) Incentives to cultivate favored minorities under alternative electoral systems. *American Political Science Review* 856–869.

Pass B (2012) Multi-marginal optimal transport and multi-agent matching problems: uniqueness and structure of solutions. *arXiv preprint arXiv:1210.7372* .

Rinott Y, Scarsini M, Yu Y (2012) A colonel blotto gladiator game. *Mathematics of Operations Research* 37(4):574–590.

Roberson B (2006) The colonel blotto game. *Economic Theory* 29(1):1–24.

Shapiro A (2001) On duality theory of conic linear problems. *Semi-infinite programming*, 135–165 (Springer).

Shubik M, Weber RJ (1981) Systems defense games: Colonel blotto, command and control. *Naval Research Logistics Quarterly* 28(2):281–287.

Thomas C (2012) N-dimensional colonel blotto game with asymmetric battlefield values. *The University of Texas at Austin Working paper*.

EC.1. Equilibrium Solutions with Two Types of Competitors

PROPOSITION EC.1. *Suppose that* $\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_2 = \mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2 = 0$, $\beta_2 = \frac{\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}{\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2} \ge \beta_1 = \frac{\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_1}{\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}$, *and* $n_i > 1$, $i = 1, 2$. *Any* (G^{1*}, G^{2*}) *with the following marginal distributions is an equilibrium.*

1. *If* $\frac{u_1 n_1}{u_2 n_2} \le \frac{\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2}{\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}$, $G_1^{1*}(z_1) = 1$ *and* $G_1^{2*}(z_1) = \left(\frac{1}{\beta_2}\cdot\frac{\lambda}{u_1}z_1^2\right)^{\frac{1}{n_2-1}}$ *for* $z_1 \in \left[0, \sqrt{\frac{u_1\beta_2}{\lambda}}\right]$ *where* $\lambda = \frac{u_1+u_2}{n_1\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2 + n_2\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2}$.

   (a) *If* $\frac{u_1}{u_2} \le \left(\frac{\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2}{\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2} - 1\right)\frac{n_2}{n_1+n_2}$, $G_2^{1*}(z_2) = L \vee \left(\frac{\lambda}{u_2}z_2^2\right)^{\frac{1}{n_1+n_2-1}}$ *and* $G_2^{2*}(z_2) = \left(\frac{\lambda}{u_2 L^{n_1}}z_2^2\right)^{\frac{1}{n_2-1}} \wedge \left(\frac{\lambda}{u_2}z_2^2\right)^{\frac{1}{n_1+n_2-1}}$ *for* $z_2 \in \left[0, \sqrt{\frac{u_2}{\lambda}}\right]$ *where* $L = \left[1 - \frac{\lambda(n_1+n_2)\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}{u_2}\right]^{\frac{1}{n_1+n_2}} \in [0,1]$.

   (b) *Otherwise,* $G_2^{1*}(z_2) = \left(\frac{\lambda}{u_2 L^{n_2}}z_2^2\right)^{\frac{1}{n_1-1}} \wedge \left(\frac{\lambda}{u_2}z_2^2\right)^{\frac{1}{n_1+n_2-1}}$ *and* $G_2^{2*}(z_2) = L \vee \left(\frac{\lambda}{u_2}z_2^2\right)^{\frac{1}{n_1+n_2-1}}$ *for* $z_2 \in \left[0, \sqrt{\frac{u_2}{\lambda}}\right]$ *where* $L = \left[\frac{n_1}{n_2}\left(\frac{\lambda(n_1+n_2)\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}{u_2} - 1\right)\right]^{\frac{1}{n_1+n_2}} \in [0,1]$.

2. *If* $\frac{u_1 n_1}{u_2 n_2} \ge \frac{\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}{\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_1}$, $G_2^{1*}(z_2) = \left(\beta_1\cdot\frac{\lambda}{u_2}z_1^2\right)^{\frac{1}{n_1-1}}$ *and* $G_2^{2*}(z_2) = 1$ *for* $z_2 \in \left[0, \sqrt{\frac{1}{\beta_1}\cdot\frac{u_2}{\lambda}}\right]$ *where* $\lambda = \frac{u_1+u_2}{n_1\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_1 + n_2\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}$.

   (a) *If* $\frac{u_2}{u_1} \le \left(\frac{\mathbf{w}_1\mathbf{D}_1^{-1}\mathbf{w}_1}{\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1} - 1\right)\frac{n_1}{n_1+n_2}$, $G_1^{1*}(z_1) = \left(\frac{\lambda}{u_1 L^{n_2}}z_1^2\right)^{\frac{1}{n_1-1}} \wedge \left(\frac{\lambda}{u_1}z_1^2\right)^{\frac{1}{n_1+n_2-1}}$ *and* $G_1^{2*}(z_1) = L \vee \left(\frac{\lambda}{u_1}z_1^2\right)^{\frac{1}{n_1+n_2-1}}$ *where* $L = \left[1 - \frac{\lambda(n_1+n_2)\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}{u_1}\right]^{\frac{1}{n_1+n_2}} \in [0,1]$.

   (b) *Otherwise,* $G_1^{1*}(z_1) = L \vee \left(\frac{\lambda}{u_1}z_1^2\right)^{\frac{1}{n_1+n_2-1}}$ *and* $G_1^{2*}(z_1) = \left(\frac{\lambda}{u_1 L^{n_1}}z_1^2\right)^{\frac{1}{n_2-1}} \wedge \left(\frac{\lambda}{u_1}z_1^2\right)^{\frac{1}{n_1+n_2-1}}$ *where* $L = \left[\frac{n_2}{n_1}\left(\frac{\lambda(n_1+n_2)\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}{u_1} - 1\right)\right]^{\frac{1}{n_1+n_2}} \in [0,1]$.

3. *Otherwise,* $G_1^{1*}(z_1) = G_2^{2*}(z_2) = 1$, $G_1^{2*}(z_1) = \left(\frac{z_1^2}{n_2\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}\right)^{\frac{1}{n_2-1}}$, *and* $G_2^{1*}(z_2) = \left(\frac{z_2^2}{n_1\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}\right)^{\frac{1}{n_1-1}}$.

Proof. The proof can be found in Section EC.2. □

PROPOSITION EC.2. *Suppose that* $\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_2 = \mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2 = 0$, $\beta_2 = \frac{\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}{\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2} \ge \beta_1 = \frac{\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_1}{\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}$, $n_i > 1$ *and* $n_{i'} = 1$. *Let* $h(L) = 1 + n_i L^{\frac{n_i+1}{n_i}} - (n_i+1)L$ *and* $C = h\left(1 - \frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\right)$. *Any* (G^{1*}, G^{2*}) *with the following marginal distributions is an equilibrium.*

1. *If* $\frac{u_{i'}}{u_i} \le \frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}n_i - \frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}\left(1 + \frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}n_i\right)\frac{n_i C}{1+n_i}$, $G_{i'}^{i'*}(z_{i'}) = 1$.

   (a) *If* $\frac{u_{i'}}{u_i} \le \left(\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i} - 1\right)\frac{n_i}{1+n_i}$, $G_{i'}^{i*}(z_{i'}) = \left(\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\cdot\frac{\lambda}{u_{i'}}z_{i'}^2\right)^{\frac{1}{n_i-1}}$ *for* $z_{i'} \in \left[0, \sqrt{\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}\frac{u_{i'}}{\lambda}}\right]$ *and* $G_i^{i*}(z_i) = \left(\frac{\lambda z_i^2}{u_i L^{n_i}}\right)^{\frac{1}{n_i-1}} \wedge \left(\frac{\lambda z_i^2}{u_i}\right)^{\frac{1}{n_i}}$, $G_i^{i'*}(z_i) = L \vee \left(\frac{\lambda z_i^2}{u_i}\right)^{\frac{1}{n_i}}$ *for* $z_i \in \left[0, \sqrt{\frac{u_i}{\lambda}}\right]$, *where* $\lambda = \frac{u_1+u_2}{n_i\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i + \mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}$ *and* $L = \left[1 - \frac{\lambda(n_i+1)\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}{u_i}\right]^{\frac{1}{n_i+1}}$.

(b) *Otherwise,* $G_{i'}^{i*}(z_{i'}) = \left(\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\cdot\frac{\lambda}{u_{i'}}z_{i'}^2\right)^{\frac{1}{n_i-1}}$ *for* $z_{i'}\in\left[0,\sqrt{\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}\cdot\frac{u_{i'}}{\lambda}}\right]$ *and* $G_i^{i*}(z_i) = \left(\frac{\lambda(1-L)}{u_i}z_i^2+L\right)^{\frac{1}{n_i}}, G_i^{i'*}(z_i) = \frac{\lambda z_i^2}{u_i}G_i^{i*}(z_i)^{1-n_i}$ *for* $z_i\in\left[0,\sqrt{\frac{u_i}{\lambda}}\right]$*, where* $\lambda = \frac{u_{i'}+u_i(1-L)}{(1-L)\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i+n_i\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}$ *and* $L\in\left[0,1-\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\right]$ *is the unique solution to*

$$\frac{u_i}{n_i+1}\frac{h(L)}{(1-L)^2}-\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i n_i u_i-\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i u_{i'}}{\left(\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i(1-L)+\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i n_i\right)n_i}=0. \tag{EC.1}$$

2. *Otherwise,*

(a) *if* $\left(1-\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}\right)\frac{u_{i'}}{u_i}\le\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}(n_i+1)-\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}\cdot\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}\left(1+\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}n_i\right)C$, $G_i^{i*}(z_i) = \left(\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\frac{\lambda z_i^2}{u_i}+1-\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\right)^{\frac{1}{n_i}}, G_i^{i'*}(z_i) = \frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}\cdot\frac{\lambda z_i^2}{u_i}G_i^{i*}(z_i)^{1-n_i}$ *for* $z_i\in\left[0,\sqrt{\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\cdot\frac{u_i}{\lambda}}\right]$ *and* $G_{i'}^{i*}(z_{i'}) = \left(\frac{\lambda z_{i'}^2}{u_{i'}L}\right)^{\frac{1}{n_i-1}}\wedge\left(\frac{\lambda z_{i'}^2}{u_{i'}}\right)^{\frac{1}{n_i}}, G_{i'}^{i'*}(z_{i'}) = L\vee\left(\frac{\lambda z_{i'}^2}{u_{i'}}\right)^{\frac{1}{n_i}}$ *for* $z_{i'}\in\left[0,\sqrt{\frac{u_{i'}}{\lambda}}\right]$*, where* $\lambda = \frac{u_{i'}+\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\cdot\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot u_i}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}+\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}n_i}$ *and* $L = \left[\frac{\left(\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}-1\right)n_i}{1+\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}n_i}-\frac{u_i}{u_{i'}}\left(\frac{\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}(n_i+1)n_i}{1+\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}n_i}-\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\cdot\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}n_iC\right)\right]^{\frac{1}{1+n_i}}\in[0,1].$

(b) *Otherwise,* $G_i^{i*}(z_i) = \left(\frac{\lambda z_i^2}{u_i}+1-\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}(1-L)\right)^{\frac{1}{n_i}}, G_i^{i'*}(z_i) = \frac{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}\cdot\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}\cdot\frac{\lambda z_i^2}{u_i(1-L)}G_i^{i*}(z_i)^{1-n_i}$*, for* $z_i\in\left[0,\sqrt{\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\cdot\frac{u_i(1-L)}{\lambda}}\right]$*, and* $G_{i'}^{i*}(z_{i'}) = \left(\frac{\lambda z_{i'}^2}{u_{i'}}+L\right)^{\frac{1}{n_i}}, G_{i'}^{i'*}(z_{i'}) = \frac{\lambda z_{i'}^2}{u_{i'}(1-L)}G_{i'}^{i*}(z_{i'})^{1-n_i}$ *for* $z_{i'}\in\left[0,\sqrt{\frac{u_{i'}(1-L)}{\lambda}}\right]$*, where* $\lambda = \frac{u_1+\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\cdot u_2}{1-L+\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}n_i}$ *and* L *is the unique solution to*

$$\frac{1}{(n_i+1)(1-L)^2}\left[\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}u_{i'}h(L)+\frac{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}u_ih\left(1-\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}(1-L)\right)\right]=\lambda. \tag{EC.2}$$

Proof. The proof can be found in Section EC.2. □

PROPOSITION EC.3. *Let* $\Gamma_i = \Gamma_i((1,v)^T)$*, defined in* (5)*. Under the conditions given in Proposition 7, the equilibrium solutions are given as follows:*

1. *If* $n_1>1$ *and* $n_2>1$*, for* $z_1\in\left[0,\sqrt{\frac{n_1\Gamma_2+n_2\Gamma_1}{\Gamma_1\Gamma_2}}\right]$*,* $i,i'=1,2$*, and* $i\ne i'$*,* $G^{i*}(z_1,vz_1) = \frac{\Gamma_1\Gamma_2z_1^2}{(n_1\Gamma_2+n_2\Gamma_1)^{\frac{n_i}{n_1+n_2}}\left[n_i(\Gamma_{i'}-\Gamma_i)^+\right]^{\frac{n_{i'}}{n_1+n_2}}}\wedge\left(\frac{\Gamma_1\Gamma_2z_1^2}{n_1\Gamma_2+n_2\Gamma_1}\right)^{\frac{1}{n_1+n_2-1}}\vee\left(\frac{n_{i'}(\Gamma_i-\Gamma_{i'})^+}{n_1\Gamma_2+n_2\Gamma_1}\right)^{\frac{1}{n_1+n_2}}.$
2. *If* $n_1=n_2=1$*, for* $z_1\in\left[0,\sqrt{\frac{2}{\Gamma_1\vee\Gamma_2}}\right]$*,* $G^{1*}(z_1,vz_1) = \frac{\Gamma_2\left(\frac{\Gamma_1}{\Gamma_2}\wedge\frac{\Gamma_2}{\Gamma_1}\right)}{2}z_1^2+\left(1-\frac{\Gamma_1}{\Gamma_2}\right)^+$*, and* $G^{2*}(z_1,vz_1) = \frac{\Gamma_1}{\Gamma_2}(G^{1*}(z_1,vz_1)-1)+1.$

3. *Otherwise, suppose that* $n_i > 1$ *and* $n_{i'} = 1$.

(a) *If* $\Gamma_i > \Gamma_{i'}$, *for* $z_1 \in \left[0, \sqrt{\frac{(1-L)\Gamma_i + n_i\Gamma_{i'}}{(1-L)\Gamma_1\Gamma_2}}\right]$, $G^{i*}(z_1, vz_1) = \left(\frac{(1-L)^2\Gamma_1\Gamma_2}{(1-L)\Gamma_i + n_i\Gamma_{i'}} z_1^2 + L\right)^{\frac{1}{n_i}}$ *and* $G^{i'*}(z_1, vz_1) = \left(\frac{(1-L)\Gamma_1\Gamma_2}{(1-L)\Gamma_i + n_i\Gamma_{i'}} z_1^2\right) G^{i*}(z_1, vz_1)^{1-n_i}$, *where* $L \in [0,1]$ *is the unique solution to*

$$\frac{\Gamma_{i'}}{(1-L)\Gamma_i + n_i\Gamma_{i'}} - \frac{1 + n_i L^{\frac{n_i+1}{n_i}} - (n_i+1)L}{(1+n_i)(1-L)^2} = 0. \tag{EC.3}$$

(b) *If* $\Gamma_i < \Gamma_{i'}$, *for* $z_1 \in \left[0, \sqrt{\frac{\Gamma_i + n_i\Gamma_{i'}}{\Gamma_1\Gamma_2}}\right]$ *and* $L = \frac{\Gamma_{i'} - \Gamma_i}{\Gamma_i + n_i\Gamma_{i'}}$, $G^{i'*}(z_1, vz_1) = \left(\frac{\Gamma_1\Gamma_2}{\Gamma_i + n_i\Gamma_{i'}} z_{i'}^2\right)^{\frac{1}{n_i}} \vee (n_i L)$, *and* $G^{i*}(z_1, vz_1) = \left(\frac{\Gamma_1\Gamma_2}{(\Gamma_i + n_i\Gamma_{i'}) n_i L} z_1^2\right)^{\frac{1}{n_i - 1}} \wedge \left(\frac{\Gamma_1\Gamma_2}{\Gamma_i + n_i\Gamma_{i'}} z_1^2\right)^{\frac{1}{n_i}}$.

Proof. The proof can be found in Section EC.2. □

EC.2. Proofs

Proof of Theorem 1.

1. The objective in (1) is rewritten as $E_{G^i}\left(\sum_{j\in\mathcal{J}} u_j H_j\left(Z_j^i\right)\right)$ where

$$H_j(z_j) = \sum_{N\subseteq\mathcal{I}_{-i}} \frac{1}{|N|+1} \prod_{\ell\in N} P_{G^{\ell*}}\left(Z_j^{\ell*} = z_j\right) \prod_{\ell\in\mathcal{I}_{-i}-N} P_{G^{\ell*}}\left(Z_j^{\ell*} < z_j\right),\ j\in\mathcal{J}, \tag{EC.4}$$

is competitor i's probability of winning event j when her score is z_j. We assume that $P_{G^{i*},G^{-i*}}\left(Z_1^{i*} = Z_1^{i'*} = z_1 \geq Z^{\ell*}, \forall \ell\in\mathcal{I}-\{i,i'\}\right) > 0$ for some z_1. Then, $\lim_{\varepsilon\downarrow 0} H_1(z_1+\varepsilon) = P_{G^{-i*}}\left(Z_1^{\ell*} \leq z_1, \forall\ell\in\mathcal{I}_{-i}\right) > H_1(z_1)$. As $\mathrm{Im}(\mathbf{W}^T)\cap\mathbb{R}_{++}^J \neq \emptyset$, we can move $\mathbf{Z}^{i*}$ along a direction $\mathbf{v}$ along which competitor i can increase her outcomes in both events, that is, $\mathbf{0} < \mathbf{v}\in\mathrm{Im}(\mathbf{W}^T)$, as

$$\mathbf{Z} = \begin{cases} \mathbf{Z}^{i*} + \varepsilon\mathbf{v}, & \text{w.p. } \frac{E\left(\Gamma_i(\mathbf{Z}^{i*})\right)}{E\left(\Gamma_i(\mathbf{Z}^{i*}+\varepsilon\mathbf{v})\right)} \wedge 1, \\ 0, & \text{w.p. } \left[1 - \frac{E\left(\Gamma_i(\mathbf{Z}^{i*})\right)}{E\left(\Gamma_i(\mathbf{Z}^{i*}+\varepsilon\mathbf{v})\right)}\right]^+, \end{cases}$$

for some $\varepsilon > 0$ small enough is feasible as $E\left(\Gamma_i(\mathbf{Z})\right) \leq E\left(\Gamma_i(\mathbf{Z}^{i*})\right) \leq 1$ and achieves a higher objective because, as $\varepsilon\to 0$,

$$\begin{aligned} & E\left(\sum_{j\in\mathcal{J}} u_j H_j(Z_j)\right) \geq \left\{\frac{E\left(\Gamma_i(\mathbf{Z}^{i*})\right)}{E\left(\Gamma_i(\mathbf{Z}^{i*}+\varepsilon\mathbf{v})\right)} \wedge 1\right\} \times \\ & \left\{E\left(\sum_{j\in\mathcal{J}} u_j H_j\left(Z_j^{i*}\right) 1_{\{Z_j^{i*}\neq z_1\}}\right) + E\left(\left(u_1 H_1(z_1+\varepsilon v_1) + \sum_{j\in\mathcal{J}_{-1}} u_2 H_2(Z_j^{i*})\right) 1_{\{Z_1^{i*}=z_1\}}\right)\right\} \\ \to\ & E\left(\sum_{j\in\mathcal{J}} u_j H_j(Z_j^{i*})\right) + u_1\left[\lim_{\varepsilon\to 0} H_1(z_1+\varepsilon) - H_1(z_1)\right] > E\left(\sum_{j\in\mathcal{J}} u_j H_j\left(Z_j^{i*}\right)\right). \end{aligned}$$

2. We can rewrite (3)–(4) as

$$\max_{G^i} \sum_{j\in\mathcal{J}} u_j P_{G^i,\mathbf{G}^{-i*}}\left(Z_j^{\ell*} \leq Z_j^i, \forall\ell\in\mathcal{I}_{-i}\right) \tag{EC.5}$$

$$\text{s.t.}\quad E_{G^i}\left(\Gamma_i(\mathbf{Z}^i)\right) \leq 1, \tag{EC.6}$$

$$P_{G^i,G^{-i*}}\left(Z_j^i = Z_j^{i'*} \geq Z_j^{\ell*}, \forall\ell\in\mathcal{I}-\{i,i'\}\right) = 0,\ \forall i'\in\mathcal{I}_{-i},. \tag{EC.7}$$

We show that the optimal objective value (EC.5) is the same with or without (EC.7) if an optimal solution to (1)–(2) exists. By applying strong duality to problem (EC.5)–(EC.6), we obtain a necessary and sufficient condition of an equilibrium.

For any optimal solution $\hat{G}^i$ to (EC.5)–(EC.6),

$$\mathbf{Z} = \begin{cases} \hat{\mathbf{Z}}^i + \varepsilon \mathbf{v} & \text{w.p. } \frac{E\left(\Gamma(\hat{\mathbf{Z}}^i)\right)}{E\left(\left(\Gamma(\hat{\mathbf{Z}}^i + \varepsilon \mathbf{v})\right)\right)} \wedge 1, \\ 0 & \text{w.p. } \left[1 - \frac{E\left(\Gamma(\hat{\mathbf{Z}}^i)\right)}{E\left(\left(\Gamma(\hat{\mathbf{Z}}^i + \varepsilon \mathbf{v})\right)\right)}\right]^+, \end{cases}$$

where $\hat{\mathbf{Z}}^i \sim \hat{G}^i$ and $\mathbf{0} < \mathbf{v} \in \mathrm{Im}(\mathbf{W}^T)$, is feasible to problem (1)–(2) with the objective value

$$\begin{aligned} E\left(\sum_{j=1}^{2} u_j H_j(Z_j)\right) &\geq \left(\frac{E\left(\Gamma(\hat{\mathbf{Z}}^i)\right)}{E\left(\left(\Gamma(\hat{\mathbf{Z}}^i + \varepsilon \mathbf{v})\right)\right)} \wedge 1\right) \times E\left(\lim_{\varepsilon \downarrow 0} \sum_{j=1}^{2} u_j H_j(\hat{Z}_j^i + \varepsilon)\right) \\ &\to \sum_{j=1}^{2} u_j P_{\hat{G}^i, \mathbf{G}^{-i*}}\left(Z_j^{\ell*} \leq \hat{Z}_j^i, \forall \ell \in \mathcal{I}_{-i}\right), \end{aligned}$$

the optimal objective value of (EC.5)–(EC.6). Therefore, the optimal objective value of (EC.5) is the same with or without (EC.7) if an optimal solution to (1)–(2) exists. Since the objective function in (EC.5) is upper semi-continuous, no duality gap exists between (EC.5)-(EC.6) and its dual

$$\min_{\lambda_i \geq 0} \left\{ \lambda_i + \max_{\mathbf{z}} \left\{ \sum_{j \in \mathcal{J}} u_j \prod_{\ell \in \mathcal{I}_{-i}} P_{G^{\ell*}}\left(Z_j^{\ell*} \leq z_j\right) - \lambda_i \Gamma_i(\mathbf{z}) \right\} \right\} \tag{EC.8}$$

by the duality Theorem of Shapiro (2001), and thus, (6) holds.

We now show that G^{i*} is an optimal solution to (3)–(4) if (6) holds and $E_{G^{i*}}\left(\Gamma_i(\mathbf{Z}^{i*})\right) \leq 1$ for all i. Note that (EC.5)–(EC.6) represent competitor i's problem in a game where *every* winner of event j gets a reward u_j. Thus,

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} u_j P_{G^{i*}, \mathbf{G}^{-i*}}\left(Z_j^{\ell*} \leq Z_j^{i*}, \forall \ell \in \mathcal{I}_{-i}\right) = \sum_{j \in \mathcal{J}} u_j E\left[\text{number of winners in event } j\right].$$

Weak duality implies that $\sum_{j \in \mathcal{J}} u_j \geq \sum_{j \in \mathcal{J}} u_j E\left[\text{number of winners in event } j\right]$. As there is always a winner in each event, there will be no tie almost surely. Consequently, the objective value at G^{i*} in (EC.5) is equal to that in (1) and G^{i*} is an optimal solution to (1)–(2).

We claim that $\lambda_i^* > 0$, because otherwise, the optimal objective value of the dual (EC.8) (and hence competitor i's expected reward) is equal to $u_1 + u_2$, which violates (6). By complementary slackness, constraint (2) must be tight and objective (EC.5) can be written as

$$\lambda_i^* + \int_{\mathbf{x}} \left\{ \sum_{j \in \mathcal{J}} u_j \prod_{\ell \in \mathcal{I}_{-i}} P_{G^{\ell*}}\left(Z_j^{\ell*} \leq z_j\right) - \lambda_i^* \Gamma_i(\mathbf{z}) \right\} dG^{i*}(\mathbf{x}),$$

which is equal to the optimal dual objective if and only if all $\mathbf{z}$ in the support of $\mathbf{Z}^{i*}$ are maximizers of the integrand. Since the set of maximizers of $\sum_{j \in \mathcal{J}} u_j \prod_{\ell \in \mathcal{I}_{-i}} P_{G^{\ell*}}\left(Z_j^{\ell*} \leq z_j\right) - \lambda_i^* \Gamma_i(\mathbf{z})$ is bounded, the support of $\mathbf{Z}^{i*}$ is also bounded.

□

Proof of Theorem 2. Suppose that an equilibrium G_1^* exists. When $\mathbf{w}_1^T\mathbf{w}_2=1$, (11) reduces to $(u_1+u_2)\left[G_1^*(z_1)\right]^{n-1}-\lambda^* z_1^2=\frac{u_1+u_2}{n}-\lambda^*$. As all the points in the support of Z_1^* are maximizers of $(u_1+u_2)\left[G_1^*(z_1)\right]^{n-1}-\lambda^* z_1^2$ and

$$(u_1+u_2)\left[G_1^*(0)\right]^{n-1}-\lambda^* 0\geq -\lambda^*\underline{z}_1^2=(u_1+u_2)\left[G_1^*(\underline{z}_1)\right]^{n-1}-\lambda^*\underline{z}_1^2,$$

where $\underline{z}_1$ is the smallest number in the bounded support, 0 must be in the support and $G_1^*(0)=0$. Thus, $\lambda^*=\frac{u_1+u_2}{n}$ and $G_1^*(z_1)=\sqrt[n-1]{\frac{z_1^2}{n}}$.

When $\mathbf{w}_1^T\mathbf{w}_2=-1$ and $n=2$, (11) reduces to $(u_1-u_2)\left[G_1^*(z_1)\right]^{n-1}-\lambda^* z_1^2=\frac{u_1-u_2}{2}-\lambda^*$. If $u_1=u_2$, then no matter what strategy a competitor applies, she will receive a reward of u_1, such that any feasible strategy is an equilibrium. If $u_1\neq u_2$, then by a similar argument as the case where $\mathbf{w}_1^T\mathbf{w}_2=1$, we have $G_1^*(z_1)=\frac{z_1^2}{2}$, $z_1\in[0,\sqrt{2}]$ when $u_1>u_2$ and $G_1^*(z_1)=1-\frac{z_1^2}{2}$, $z_1\in[-\sqrt{2},0]$ when $u_2>u_1$.

When $\mathbf{w}_1^T\mathbf{w}_2=-1$, (11) reduces to $u_1G_1^{*n-1}(z_1)+u_2(1-G_1^*(z_1))^{n-1}=\lambda^* z_1^2+\frac{u_1+u_2}{n}-\lambda^*$, which we claim has a unique solution G_1^*. The right hand side is a convex function of z_1 that achieves its minimum at $z_1=0$, while the left hand side first decreases until it reaches $z_1=G_1^{*-1}(t_0)$ and then increases when $n>2$. Thus, $G_1^{*-1}(t_0)=0$ and $u_1\left[G_1^*(0)\right]^{n-1}+u_2\left[1-G_1^*(0)\right]^{n-1}=\frac{u_1+u_2}{n}-\lambda^*$, i.e., $\lambda^*=\frac{u_1+u_2}{n}-u_1t_0^{n-1}-u_2(1-t_0)^{n-1}$ when $n>2$.

We can verify that the solutions derived above satisfy (11), and thus are the unique equilibria under the three cases.

□

Proof of Proposition 1.

1. When $\mathbf{w}_1+\mathbf{w}_2=0$, i.e., $\mathbf{w}_1=-\mathbf{w}_2$, suppose that $P_{G^{i*},G^{-i*}}\left(Z_1^{i*}=Z_1^{i'*}=Z_1^{i''*}=z_1\right)>0$ for some z_1. Then, competitor i's total reward $u_1H_1\left(Z_1^{i*}\right)+u_2H_2\left(-Z_1^{i*}\right)$ is discontinuous at $Z_1^{i*}=z_1$ as

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\{u_1H_1(z_1+\varepsilon)+u_2H_2(-(z_1+\varepsilon))+u_1H_1(z_1-\varepsilon)+u_2H_2(-(z_1-\varepsilon))-2(u_1H_1(z_1)+u_2H_2(-z_1))\}\\ =&\sum_{\emptyset\neq N\subseteq\mathcal{I}_{-i}}\frac{|N|-1}{|N|+1}\left[u_1\prod_{\ell\in\mathcal{I}_{-i}-N}P_{G^{\ell*}}\left(Z^{\ell*}<z_1\right)+u_2\prod_{\ell\in\mathcal{I}_{-i}-N}P_{F_\ell^*}\left(Z^{\ell*}>z_1\right)\right]\prod_{\ell\in N}P_{F_\ell^*}\left(Z^{\ell*}=z_1\right)>0.\end{aligned}$$

Without loss of generality, we assume that $\lim_{\varepsilon\downarrow 0}\{u_1H_1(z_1+\varepsilon)+u_2H_2(-(z_1+\varepsilon))\}>u_1H_1(z_1)+u_2H_2(-z_1)$. When $z_1\neq 0$, we move $\mathbf{Z}^{i*}$ along $(1,-1)^T$ at $Z_1^{i*}=z_1$ as

$$\mathbf{Z}=\begin{cases}\mathbf{Z}^{i*}, & \text{if } Z_1^{i*}\neq z_1,\\ \left.\begin{array}{ll}\mathbf{Z}^{i*}+\varepsilon(1,-1)^T, & \text{w.p. } \frac{z_1^2}{(z_1+\epsilon)^2}\wedge 1,\\ 0, & \text{w.p. } \left[1-\frac{z_1^2}{(z_1+\epsilon)^2}\right]^+,\end{array}\right\} & \text{if } Z_1^{i*}=z_1,\end{cases}$$

and move $\mathbf{Z}^{i*}$ along $(1,-1)^T$ at every point when $z_1 = 0$ as

$$\mathbf{Z} = \begin{cases} \mathbf{Z}^{i*} + \varepsilon(1,-1)^T, & \text{w.p. } \frac{E\left(\Gamma_i(\mathbf{Z}^{i*})\right)}{E\left(\Gamma_i(\mathbf{Z}^{i*}+\varepsilon(1,-1)^T)\right)} \wedge 1, \\ 0, & \text{w.p. } \left[1 - \frac{E\left(\Gamma_i(\mathbf{Z}^{i*})\right)}{E\left(\Gamma_i(\mathbf{Z}^{i*}+\varepsilon(1,-1)^T)\right)}\right]^+. \end{cases}$$

We can easily verify that $\mathbf{Z}$ is feasible and, when $z_1 \neq 0$, yields a higher objective value than $\mathbf{Z}^{i*}$. Thus, $u_1 H_1(Z_1^{i*}) + u_2 H_2(-Z_1^{i*})$ must be continuous at z_1 if $P_{F_i^*}(Z_1^{i*} = z_1 \neq 0) > 0$ and, as $\varepsilon \to 0$, the expected total reward under $\mathbf{Z}$ when $z_1 = 0$,

$$\begin{aligned} E\left(\sum_{j=1}^{2} u_j H_j(Z_j)\right) &\geq \left\{\frac{E\left(\Gamma_i(\mathbf{Z}^{i*})\right)}{E\left(\Gamma_i(\mathbf{Z}^{i*}+\varepsilon(1,-1)^T)\right)}\right\} \times \left\{E\left[\left(u_1 H_1\left(Z_1^{i*}+\varepsilon\right) + u_2 H_2\left(-(Z_1^{i*}+\varepsilon)\right)\right)\right]\right\} \\ \to E\left(\sum_{j=1}^{2} u_j H_j(Z^{i*})\right) &+ \left[\lim_{\varepsilon\to 0} u_1 H_1(\varepsilon) + u_2 H_2(-\epsilon) - u_1 H_1(0) - u_2 H_2(0)\right] > E\left(\sum_{j=1}^{2} u_j H_j\left(Z_j^{i*}\right)\right), \end{aligned}$$

the expected total reward under $\mathbf{Z}^{i*}$.

2. It follows from the same sufficiency argument as in the proof of Theorem 1.

□

Proof of Proposition 2. If $\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_2 = 0$, any feasible solution to the left hand side of (8) is optimal and is therefore an equilibrium. If $\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_2 > 0$ and $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent, $Z_1^{i*} = Z_2^{i*}$ and thus $G^{i*}(\mathbf{z}) = G_1^{i*}(z_1) \wedge G_2^{i*}(z_2)$. If $\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_2 > 0$ and $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent, then the optimal solutions to (8) are the same as those to

$$\min_{G^i} \left\{ -\int_{\mathbf{z}} z_1 z_2 dG^i(\mathbf{z}) \middle| G^i \text{ has marginal distributions } G_1^{i*}, G_2^{i*} \right\}, \tag{EC.9}$$

whose dual is given by

$$\max_{\psi_1,\psi_2} \left\{ \sum_{j=1}^{2} \int_{z_j} \psi_j(z_j) dG_j^{i*}(z_j) \middle| \sum_{j=1}^{2} \psi_j(z_j) \leq -z_1 z_2 \right\}. \tag{EC.10}$$

Let $\underline{z}_j$ and $\bar{z}_j$ be the smallest and largest numbers on the support of Z_j^{i*} and define

$$\begin{aligned} \psi_1^*(z_1) &= -\underline{z}_1\underline{z}_2 - \int_{\underline{z}_1}^{z_1} \underline{z}_2 \vee \sup\left\{z_2 : G_2^{i*}(z_2) < G_1^{i*}(u)\right\} du, \\ \psi_2^*(z_2) &= -\int_{\underline{z}_2}^{z_2} \bar{z}_1 \wedge \inf\left\{z_1 : G_1^{i*}(z_1) > G_2^{i*}(u)\right\} du. \end{aligned}$$

As $\inf\{z_1 : G_1^{i*}(z_1) > G_2^{i*}(u)\} = \inf\{z_1 : \sup\{z_2 : G_2^{i*}(z_2) < G_1^{i*}(z_1)\} > u\}$, $\psi_1^*(z_1) + \psi_2^*(z_2) \leq -z_1 z_2$ and the equality holds if and only if $G_1^{i*}(z_1) > G_2^{i*}(z_2) \geq \lim_{\varepsilon\downarrow 0} G_1^{i*}(z_1 - \varepsilon)$ or $\forall\delta > 0$, $G_2^{i*}(z_2 - \delta) < G_1^{i*}(z_1) \leq G_2^{i*}(z_2)$ by Young's inequality (Mitroi and Niculescu (2011)). Thus, (ψ_1^*, ψ_2^*) is a feasible solution to (EC.10) and $\psi_1^*(z_1) + \psi_2^*(z_2) = -z_1 z_2$ on the support of $G_1^{i*}(z_1) \wedge G_2^{i*}(z_2)$, that is, no duality gap exists. Thus, $G^{i*}(\mathbf{z}) = G_1^{i*}(z_1) \wedge G_2^{i*}(z_2)$ is the unique optimal solution to (EC.9) by complementary slackness.

If $\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_2 < 0$ and $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent, $Z_2^{i*} = -Z_1^{i*}$ and thus, $G^{i*}(\mathbf{z}) = [G_1^{i*}(z_1) + G_2^{i*}(z_2) - 1]^+$. If $\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_2 < 0$ and $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent, the result follows from a similar argument as the case when $\mathbf{w}_1^T \mathbf{D}_i^{-1} \mathbf{w}_2 > 0$ and $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent with $z_1 z_2$ replaced by $-z_1 z_2$ and

$$\psi_1^*(z_1) = \int_{\underline{z}_1}^{z_1} \underline{z}_2 \vee \sup\left\{z_2 : G_1^{i*}(u) + G_2^{i*}(z_2) < 1\right\} du,$$
$$\psi_2^*(z_2) = \underline{z}_1 \bar{z}_2 + \int_{\bar{z}_2}^{z_2} \underline{z}_1 \vee \sup\left\{z_1 : G_1^{i*}(z_1) + G_2^{i*}(u) < 1\right\} du.$$

□

Proof of Proposition 3. We can easily check that a distribution with the given marginals satisfies (6) with $\lambda^* = \frac{u_1+u_2}{n}$ and therefore, is an equilibrium by Theorem 1. For any equilibrium G^*, the support of Z_j^* must be an interval with left end 0 by Theorem 1. Solving (9) with the boundary condition $G_j^*(0) = 0$, we obtain $G_j^*(z_j) = \left(\frac{u_1+u_2}{nu_j} z_j^2\right)^{\frac{1}{n-1}}$.

□

We first establish Lemma EC.1 which is critical in the proof of Proposition 4.

LEMMA EC.1. *There is a unique λ^*, $\lambda^* > \frac{u_1+u_2}{n} - u_1 t_0^{n-1} - u_2(1-t_0)^{n-1}$ where $t_0 \in [0,1]$ is the maximizer of $u_1 t^{n-1} + u_2(1-t)^{n-1}$, such that a solution to* (EC.17) *satisfying $\lim_{t\downarrow 0}\theta(t) = -\lim_{t\uparrow 1}\theta(t) = -\theta_0$ exists. Furthermore, such a solution is unique.*

Proof. First, we show that for each $\lambda^* > 0$, a unique solution exists, denoted as $\theta_\infty(t)$, to (EC.17) such that $\lim_{t\downarrow 0}\theta_\infty(t) = -\theta_0$. As, for any $(t,\theta) \in [0,1] \times [-\theta_0, \theta_0] \setminus \{(0,-\theta_0),(1,\theta_0)\}$, the right hand side of (EC.17) is locally Lipschitz in θ (or its reciprocal is locally Lipschitz in t), a unique solution $\theta(\cdot)$ to (EC.17) exists such that $\theta(t) = \theta$ by Picard's Theorem. Let $\theta_k(\cdot)$, $k = 1, 2, \ldots$, be the unique solution to (EC.17) passing through $(t,\theta) = \left(0, \left(-1+\frac{1}{k}\right)\theta_0\right)$. Then, $\theta_k(t)$ is decreasing in k and thus, point-wise converges to a function which we will show is the unique solution $\theta_\infty(t)$ to (EC.17) such that $\lim_{t\downarrow 0}\theta_\infty(t) = -\theta_0$.

For $t \in [\varepsilon, 2\varepsilon]$, $\varepsilon > 0$ small enough, $\theta_k(t)$ is bounded from below by the solution to (EC.17) passing through $\left(\frac{\varepsilon}{2}, -\theta_0\right)$. Thus, $\theta_k'(t)$ is uniformly convergent as $k \to \infty$. As $\limsup_{t\downarrow 0}\theta_\infty(t) \le \lim_{t\downarrow 0}\theta_k(t) = \left(-1+\frac{1}{k}\right)\theta_0$ for all k, $\lim_{t\downarrow 0}\theta_\infty(t) = -\theta_0$ and $\theta_\infty(t)$ is a solution.

Any solution to (EC.17) starts with $(t,\theta) = (0,-\theta_0)$ and will never cross other solutions even if they exist. As the right hand side of (EC.17) is decreasing in θ when t and θ is small enough, θ_∞ is the unique solution.

We then show that there exists a unique λ^*, $\lambda^* > \frac{u_1+u_2}{n} - u_1 t_0^{n-1} - u_2(1-t_0)^{n-1}$, at which $\theta_\infty(1) = \theta_0$. By (EC.17),

$$\frac{d\theta}{dt} \le \frac{(n-1)(u_1+u_2)\cot(\theta+\theta_0)}{\lambda^*}$$

for $\theta < 0$. As $\frac{d\theta}{dt} = \frac{(n-1)(u_1+u_2)\cot(\theta+\theta_0))}{\lambda^*}$, where $\theta(0) = -\theta_0$ has a solution $\cos(\theta(t) + \theta_0) = e^{-\frac{(n-1)(u_1+u_2)}{\lambda^*}t}$, $\theta(1) < 0$ and $\theta_\infty(t)$ intersects with the line $t = 1$ for λ^* that is large enough.

Let $\lambda^*(\varepsilon) = \frac{u_1+u_2}{n} - u_1 t_0^{n-1} - u_2(1-t_0)^{n-1} + \varepsilon$. For $t \in [t_0 - \delta, t_0 + \delta]$, where $\delta > 0$ small enough,

$$\theta(t_0+\delta) \geq -\theta_0 + \min_{t\in[t_0-\delta,t_0+\delta]} \theta'_\infty(t)2\delta$$

by mean-value Theorem, and

$$\lim_{\varepsilon\downarrow 0} \min_{t\in[t_0-\delta,t_0+\delta]} \theta'_\infty(t)2\delta \geq \lim_{\varepsilon\downarrow 0} \frac{n-1}{2} \frac{\min\limits_{t\in[t_0-\delta,t_0+\delta],\theta\in[-\theta_0,\theta_0]} \left\{u_1 t^{n-2}\cot(\theta+\theta_0) + u_2(1-t)^{n-2}\cot(\theta_0-\theta)\right\}}{\max\limits_{t\in[t_0-\delta,t_0+\delta]} \left\{u_1 t_0^{n-1} + u_2(1-t)^{n-1} - \frac{u_1+u_2}{n} + \lambda^*(\varepsilon)\right\}} 2\delta$$
$$= \frac{\cot(\theta_0)\left[u_1 t_0^{n-2} + u_2(1-t_0)^{n-2}\right] + O(\delta)}{O(\delta^2)}\delta \to \infty, \text{ as } \delta \to 0.$$

Thus, there exist $\varepsilon, \delta > 0$ such that $\theta_\infty(t) = \theta_0$ at $\lambda^* = \lambda^*(\varepsilon)$ for some $t \leq t_0 + \delta$. Thus, for λ^* that is small enough, $\theta_\infty(t)$ intersects with the line $\theta = \theta_0$.

Since $\theta_\infty(t)$ and hence its unique intersection with $t = 1$ or $\theta = \theta_0$ are continuous in λ^*, a λ^* exists such that $\theta_\infty(1) = \theta_0$. Furthermore, it can be shown by contradiction that $\theta_\infty(t)$ passes through $(1, \theta_0)$ only if $\theta'_\infty(t) \geq 0$ for all t by (EC.17). Thus, $\theta'_\infty(t)$ is decreasing in λ^* and there exists a unique λ^* such that $\theta_\infty(1) = \theta_0$.

□

Proof of Proposition 4.

1. When $\mathbf{w}_1^T\mathbf{w}_2 > 0$, $G_1^*(z_1(t)) = G^*(\mathbf{z}(t)) = G_2^*(z_2(t))$ and $\mathbf{z}(0) = \mathbf{0}$ by Theorem 1 as

$$\sum_{j=1}^{2} u_j \left[G_j^*(z_j(0))\right]^{n-1} - \lambda \mathbf{z}(0)^T(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}(0) = -\lambda\mathbf{z}(0)^T(WW^T)^{-1}\mathbf{z}(0) \leq \sum_{j=1}^{2} u_j \left[G_j^*(0)\right]^{n-1}.$$

We then establish that, when $\mathbf{z}(t)$ is increasing in t, an equilibrium must be unique if it exists. The first-order condition (9) is equivalent to

$$u_j(n-1)\left[G_j^*(z_j(t))\right]^{n-2} G_j^{*'}(z_j(t))z'_j(t) = \frac{2\lambda^*}{1-(\mathbf{w}_1^T\mathbf{w}_2)^2}(z_j(t) - \mathbf{w}_1^T\mathbf{w}_2 z_{3-j}(t))z'_j(t). \quad \text{(EC.11)}$$

Since $G_1^*(z_1(t)) = G_2^*(z_2(t))$, (EC.11) reduces to

$$\frac{dz_2}{dz_1} = \frac{u_2(z_1 - z_2\mathbf{w}_1^T\mathbf{w}_2)}{u_1(z_2 - z_1\mathbf{w}_1^T\mathbf{w}_2)} \geq 0, \quad \text{(EC.12)}$$

as $\mathbf{z}(t)$ is increasing in t. Letting $v = \frac{z_2}{z_1}$, we can rewrite (EC.12) as

$$z_1\frac{dv}{dz_1} = \frac{u_2 - (u_2-u_1)\mathbf{w}_1^T\mathbf{w}_2 v - u_1 v^2}{u_1(v - \mathbf{w}_1^T\mathbf{w}_2)}. \quad \text{(EC.13)}$$

For any v that satisfies the boundary condition, $u_2 - (u_2 - u_1)\mathbf{w}_1^T\mathbf{w}_2 v - u_1 v^2 = 0$, which has exactly two solutions: one positive and one negative. Since $\frac{dz_2}{dz_1} = v + \frac{dv}{dz_1}z_1 \geq 0$, only the positive solution, $v = \frac{\sqrt{[(u_2-u_1)\mathbf{w}_1^T\mathbf{w}_2]^2 + 4u_1u_2} - (u_2-u_1)\mathbf{w}_1^T\mathbf{w}_2}{2u_1}$, satisfies (EC.12) and a unique solution $G_1^*(z_1) =$

$\left(\frac{\lambda^*(1-\mathbf{w}_1^T\mathbf{w}_2 v)}{u_1\left(1-\left(\mathbf{w}_1^T\mathbf{w}_2\right)^2\right)}z_1^2\right)^{\frac{1}{n-1}}$ to (EC.11) exists for any $\lambda^* \geq 0$. The support of G^* is the line segment $\left\{(z_1, vz_1)^T \middle| z_1 \in \left[0, G_1^{*-1}(1)\right]\right\}$. By Theorem 1, (6) becomes $\lambda^* + (u_1+u_2)G_1^{*n-1}(z_1(t)) - \lambda^*\mathbf{z}(t)^T(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}(t) = \frac{u_1+u_2}{n}$ and thus, $\lambda^* = \frac{u_1+u_2}{n}$ and is unique.

As the Hessian of $\sum_{j=1}^{2} u_j G_j^{*n-1}(z_j) - \lambda^*\mathbf{z}^T(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}$, $\frac{2\mathbf{w}_1^T\mathbf{w}_2\lambda^*}{1-\left(\mathbf{w}_1^T\mathbf{w}_2\right)^2}\begin{pmatrix} -v & 1 \\ 1 & -1/v \end{pmatrix}$ where v is the positive root of $u_2 - (u_2-u_1)\mathbf{w}_1^T\mathbf{w}_2 v - u_1 v^2 = 0$, is negative semi-definite and an equilibrium exists.

2. When $\mathbf{w}_1^T\mathbf{w}_2 < 0$, $G_1^*(z_1(t)) + G_2^*(z_2(t)) = 1$ for any given $t \in [0,1]$. Letting $\hat{\mathbf{z}} = z_2(0)(\mathbf{w}_1^T\mathbf{w}_2, 1)^T$, we have

$$\sum_{j=1}^{2}\left[G_j^*(\hat{z}_j)\right]^{n-1} - \lambda\hat{\mathbf{z}}^T(\mathbf{W}^T\mathbf{W})^{-1}\hat{\mathbf{z}} \geq \sum_{j=1}^{2}\left[G_j^*(z_j(0))\right]^{n-1} - \lambda\mathbf{z}(0)^T(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}(0)$$

and hence, $\mathbf{z}(0) = \hat{\mathbf{z}}$ by Theorem 1. Following a similarly argument, $z_2(1) = z_1(1)\mathbf{w}_1^T\mathbf{w}_2$ and the support of $\mathbf{Z}^*$ intersects with the lines $z_1 = z_2\mathbf{w}_1^T\mathbf{w}_2$ and $z_2 = z_1\mathbf{w}_1^T\mathbf{w}_2$ at $\mathbf{z}(0)$ and $\mathbf{z}(1)$, respectively.

We now establish that, when $\mathbf{w}_1^T\mathbf{w}_2 < 0$, an equilibrium must be unique if it exists. Then, the first-order condition can be simplified as

$$(n-1)u_1 t^{n-2} = \frac{2\lambda^*}{1-\left(\mathbf{w}_1^T\mathbf{w}_2\right)^2}(z_1(t) - z_2(t)\mathbf{w}_1^T\mathbf{w}_2)z_1'(t), \tag{EC.14}$$

$$-(n-1)u_2(1-t)^{n-2} = \frac{2\lambda^*}{1-\left(\mathbf{w}_1^T\mathbf{w}_2\right)^2}(z_2(t) - z_1(t)\mathbf{w}_1^T\mathbf{w}_2)z_2'(t). \tag{EC.15}$$

Letting $r(t) = \sqrt{\mathbf{z}(t)(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}(t)}$ and $\theta(t) = \arcsin\left(\frac{z_2(t)-z_1(t)}{2r(t)\sin(\theta_0)}\right)$
where $\theta_0 = \arcsin\left(\sqrt{\frac{1-\mathbf{w}_1^T\mathbf{w}_2}{2}}\right)$ and applying (6), we have the equivalence between (EC.14)–(EC.15) and

$$r^2(t) = \frac{1}{\lambda^*}\left(u_1 t^{n-1} + u_2(1-t)^{n-1} - \frac{u_1+u_2}{n} + \lambda^*\right), \tag{EC.16}$$

$$\frac{1}{n-1}\frac{d\theta}{dt} = \frac{u_1 t^{n-2}\cot(\theta+\theta_0) + u_2(1-t)^{n-2}\cot(\theta_0-\theta)}{2\left(u_1 t^{n-1} + u_2(1-t)^{n-1} - \frac{u_1+u_2}{n} + \lambda^*\right)}. \tag{EC.17}$$

The differential equation (EC.17) has the boundary conditions $\theta_0 = \theta(1) = -\theta(0)$ and has a unique solution $(\lambda^*, \theta(t))$ by Lemma EC.1.

An equilibrium exists if and only if the Hessian of $\sum_{j=1}^{2} u_j G_j^*(z_j) - \lambda^*\mathbf{z}\left(\mathbf{W}^T\mathbf{W}\right)^{-1}\mathbf{z}$ at $\mathbf{z}(t)$,

$$\frac{\mathbf{w}_1^T\mathbf{w}_2(n-1)^2 u_1 u_2 t^{n-2}(1-t)^{n-2}}{2\lambda r^2(t)\sin(\theta(t)+\theta_0)\sin(\theta_0-\theta(t))} \cdot \begin{pmatrix} \frac{1}{\left(z_1'(t)\right)^2} & -\frac{1}{z_1'(t)z_2'(t)} \\ -\frac{1}{z_1'(t)z_2'(t)} & \frac{1}{\left(z_2'(t)\right)^2} \end{pmatrix}$$

is negative semi-definite, which holds as $\mathbf{w}_1^T\mathbf{w}_2 < 0$.

□

Proof of Proposition 5. Let

$$\hat{\mathbf{z}}(t) = \begin{cases} \hat{r}(t)(\mathbf{w}_1^T\mathbf{w}_2, 1), & \text{when } t \in [0, t_0], \\ \hat{r}(t)(1, \mathbf{w}_1^T\mathbf{w}_2), & \text{when } t \in [t_0, 1], \end{cases}$$

and $\hat{G}_1(\hat{z}_1(t)) = 1 - \hat{G}_2(\hat{z}_2(t)) = t$, where $\hat{r}(t) = \sqrt{\frac{u_1 t^{n-1} + u_2(1-t)^{n-1}}{\frac{u_1+u_2}{n} - u_1 t_0^{n-1} - u_2(1-t_0)^{n-1}}}$. If all other competitors are applying $\hat{G}$, then the optimization problem for an individual competitor is

$$\max_G \int_{\mathbf{z}} \sum_{j=1}^{2} u_j \hat{G}_j^{n-1}(z_j) dG(\mathbf{z}) \tag{EC.18}$$

$$\text{s.t.} \int_{\mathbf{z}} \mathbf{z}(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z} dG(\mathbf{z}) \le 1. \tag{EC.19}$$

The objective value of $\hat{G}$ is $\frac{u_1+u_2}{n}$, while the optimal objective value can be bounded by the dual objective $\lambda + \max_{\mathbf{z}} \left\{ \sum_{j=1}^{2} u_j \hat{G}_j(z_j) - \lambda \mathbf{z} \left(\mathbf{W}^T\mathbf{W}\right)^{-1} \mathbf{z} \right\}$ for any feasible $\lambda > 0$. Taking $\lambda = \frac{u_1+u_2}{n} - u_1 t_0^{n-1} - u_2(1-t_0)^{n-1} > 0$, we have

$$\begin{aligned}
&\lambda + \max_{\mathbf{z}} \left\{ \sum_{j=1}^{2} u_j \hat{G}_j(z_j) - \lambda \mathbf{z} \left(\mathbf{W}^T\mathbf{W}\right)^{-1} \mathbf{z} \right\} \\
=&\lambda + \max_{t_1, t_2 \in [0,1]} \left\{ u_1 t_1^{n-1} + u_2(1-t_2)^{n-1} - \lambda(\hat{z}_1(t_1), \hat{z}_2(t_2)) \left(\mathbf{W}^T\mathbf{W}\right)^{-1} \begin{pmatrix} \hat{z}_1(t_1) \\ \hat{z}_2(t_2) \end{pmatrix} \right\} \\
\le& \frac{u_1+u_2}{n} + u_1 t_0^{n-1} \vee u_2(1-t_0)^{n-1}.
\end{aligned}$$

Therefore, the difference in objective value (EC.18) between the optimal solution and $\hat{G}$ is upper bounded by $u_1 t_0^{n-1} \vee u_2(1-t_0)^{n-1}$.

Recall that $(\lambda^*, \theta(t))$ is the unique solution to (EC.17) with the boundary condition $\theta(0) = -\theta(1) = -\theta_0$. Also note that $r(t) = \sqrt{\frac{u_1 t^{n-1} + u_2(1-t)^{n-1} - \frac{u_1+u_2}{n} + \lambda^*}{\lambda^*}}$, $\hat{r}(t) = \sqrt{\frac{u_1 t^{n-1} + u_2(1-t)^{n-1}}{\hat{\lambda}}}$ where $\hat{\lambda} = \frac{u_1+u_2}{n} - u_1 t_0^{n-1} - u_2(1-t_0)^{n-1} = \frac{u_1+u_2}{n} + O(2^{-n})$. Let $\hat{\theta}(t) = -\theta_0$ when $t \in [0, t_0]$ and $\hat{\theta}(t) = \theta_0$ when $t \in (t_0, 1]$.

As $\theta'(t) \ge 0$, we have

$$\begin{aligned}
t &\ge \frac{\sqrt[n-2]{u_2 \cot(\theta(t) - \theta_0)}}{\sqrt[n-2]{u_1 \cot(\theta(t) + \theta_0)} + \sqrt[n-2]{u_2 \cot(\theta(t) - \theta_0)}} \text{ for } t \in \left[0, \frac{\sqrt[n-2]{u_2 \cot\left(-\frac{\pi}{4} - \theta_0\right)}}{\sqrt[n-2]{u_1 \cot\left(\theta_0 - \frac{\pi}{4}\right)} + \sqrt[n-2]{u_2 \cot\left(-\frac{\pi}{4} - \theta_0\right)}}\right], \\
t &\le \frac{\sqrt[n-2]{u_2 \cot(\theta(t) - \theta_0)}}{\sqrt[n-2]{u_1 \cot(\theta(t) + \theta_0)} + \sqrt[n-2]{u_2 \cot(\theta(t) - \theta_0)}} \text{ for } t \in \left[\frac{\sqrt[n-2]{u_2 \cot\left(\frac{\pi}{4} - \theta_0\right)}}{\sqrt[n-2]{u_1 \cot\left(\theta_0 + \frac{\pi}{4}\right)} + \sqrt[n-2]{u_2 \cot\left(\frac{\pi}{4} - \theta_0\right)}}, 1\right].
\end{aligned}$$

Thus, $\theta(t) + \theta_0 = O(3^{-n})$ for $t \in \left[0, \frac{1}{4}\right]$ and $\theta_0 - \theta(t) = O(3^{-n})$ for $t \in \left[\frac{3}{4}, 1\right]$. Furthermore, according to the mean value theorem,

$$\frac{\frac{\pi}{8} - \left(-\frac{\pi}{8}\right)}{\frac{\sqrt[n-2]{u_2 \cot\left(\frac{\pi}{4} - \theta_0\right)}}{\sqrt[n-2]{u_1 \cot\left(\theta_0 + \frac{\pi}{4}\right)} + \sqrt[n-2]{u_2 \cot\left(\frac{\pi}{4} - \theta_0\right)}} - \frac{\sqrt[n-2]{u_2 \cot\left(-\frac{\pi}{4} - \theta_0\right)}}{\sqrt[n-2]{u_1 \cot\left(\theta_0 - \frac{\pi}{4}\right)} + \sqrt[n-2]{u_2 \cot\left(-\frac{\pi}{4} - \theta_0\right)}}} \le \frac{(n-1)(u_1 t_0^{n-2} + u_2(1-t_0)^{n-2})O(1)}{\lambda^* - \hat{\lambda}},$$

and thus, $\lambda^* - \hat{\lambda} = O\left(u_1 t_0^{n-2} + u_2(1-t_0)^{n-2}\right) = O(2^{-n})$. Therefore,

$$\begin{aligned}
&\sup_{t\in\left[0,\frac{1}{4}\right]} \left\|(r(t)\cos\theta(t), r(t)\sin\theta(t)) - \left(\hat{r}(t)\cos\hat{\theta}(t), \hat{r}(t)\sin\hat{\theta}(t)\right)\right\|_2 \\
&\leq O(\hat{r}(0)(\theta(t)+\theta_0)) + \sup_{t\in\left[0,\frac{1}{4}\right]} |r(t) - \hat{r}(t)| = O\left(\frac{\sqrt{n}}{3^n}\right), \\
&\sup_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \left\|(r(t)\cos\theta(t), r(t)\sin\theta(t)) - \left(\hat{r}(t)\cos\hat{\theta}(t), \hat{r}(t)\sin\hat{\theta}(t)\right)\right\|_2 \\
&\leq \sup_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} r(t) + \hat{r}(t) = O\left(\sqrt{n}\left(\frac{\sqrt{3}}{2}\right)^n\right), \\
&\sup_{t\in\left[\frac{3}{4},1\right]} \left\|(r(t)\cos\theta(t), r(t)\sin\theta(t)) - \left(\hat{r}(t)\cos\hat{\theta}(t), \hat{r}(t)\sin\hat{\theta}(t)\right)\right\|_2 \\
&\leq O(\hat{r}(1)(\theta_0 - \theta(t))) + \sup_{t\in\left[\frac{3}{4},1\right]} |r(t) - \hat{r}(t)| = O\left(\frac{\sqrt{n}}{3^n}\right),
\end{aligned}$$

and hence, $\sup_{t\in[0,1]} \left\|(r(t)\cos\theta(t), r(t)\sin\theta(t)) - \left(\hat{r}(t)\cos\hat{\theta}(t), \hat{r}(t)\sin\hat{\theta}(t)\right)\right\| \to 0$ as $n\to\infty$.

□

Proof of Proposition EC.1. If $\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_2 = \mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_2 = 0$, then, $\mathbf{z}(\mathbf{W}^T\mathbf{D}_i^{-1}\mathbf{W})^{-1}\mathbf{z} = \frac{z_1^2}{\mathbf{w}_1^T\mathbf{D}_i^{-1}\mathbf{w}_1} + \frac{z_2^2}{\mathbf{w}_2^T\mathbf{D}_i^{-1}\mathbf{w}_2}$, $i=1,2$.

1. We show that (12) holds at $(\lambda_1^*, \lambda_2^*) = (\lambda\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2, \lambda\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2)$.
   (a) Both G^{1*} and G^{2*} are feasible as

$$\begin{aligned}
\int \mathbf{z}(\mathbf{W}^T\mathbf{D}_1^{-1}\mathbf{W})^{-1}\mathbf{z}dG^{1*}(\mathbf{z}) &= \int_L^1 \frac{u_2}{\lambda_1^*} d\frac{(G_1^{1*})^{n_1+n_2}}{n_1+n_2} = \frac{u_2(1-L^{n_1+n_2})}{\lambda_1^*(n_1+n_2)} = 1, \\
\int \mathbf{z}(\mathbf{W}^T\mathbf{D}_2^{-1}\mathbf{W})^{-1}\mathbf{z}dG^{2*}(\mathbf{z}) &= \frac{1}{\lambda_2^*}\left(\int_0^1 u_1 d\frac{(G_1^{2*})^{n_2}}{n_2} + \int_0^L u_2 L^{n_1} d\frac{(G_2^{2*})^{n_2}}{n_2} + \int_L^1 u_2 d\frac{(G_2^{2*})^{n_1+n_2}}{n_1+n_2}\right) \\
&= \frac{1}{\lambda_2^*}\left[\frac{u_1}{n_2} + \frac{u_2}{n_1+n_2}\left(1 + \frac{n_1 L^{n_1+n_2}}{n_2}\right)\right] = 1.
\end{aligned}$$

Since $\beta_1 \leq \beta_2$, for $z_1 \in \left[0, \sqrt{\frac{u_1\beta_2}{\lambda}}\right]$ and $z_2 \in \left[0, \sqrt{\frac{u_2}{\lambda}}\right]$,

$$\begin{aligned}
\sum_{j=1}^2 u_j G_j^{1*}(z_j)^{n_1-1} G_j^{2*}(z_j)^{n_2} &= \lambda_1^* \frac{z_1^2}{\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_1}\left[\frac{\beta_1}{\beta_2}\left(\frac{\lambda z_1^2}{u_1\beta_2}\right)^{\frac{1}{n_2-1}}\right] + \lambda_1^* \frac{z_2^2}{\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}\left[\frac{1}{L}\left(\frac{\lambda_1^* z_2^2}{u_2 L^{n_1}}\right)^{\frac{1}{n_2-1}} \wedge 1\right] \\
&\leq \lambda_1^*\left(\frac{z_1^2}{\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_1} + \frac{z_2^2}{\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}\right), \\
\sum_{j=1}^2 u_j G_j^{1*}(z_j)^{n_1} G_j^{2*}(z_j)^{n_2-1} &= \lambda_2^*\left(\frac{z_1^2}{\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1} + \frac{z_2^2}{\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2}\right),
\end{aligned}$$

and the equality holds when $z_1 = z_2 = 0$.
   (b) Both G^{1*} and G^{2*} are feasible as

$$\int \mathbf{z}(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}dG^{1*}(\mathbf{z}) = \frac{u_2}{\lambda_1^*}\left(\int_0^L L^{n_2} d\frac{(G_1^{1*})^{n_1}}{n_1} + \int_L^1 d\frac{(G_1^{1*})^{n_1+n_2}}{n_1+n_2}\right)$$

$$
\begin{aligned}
&= \frac{u_2}{\lambda_1^*}\left(\frac{n_1+n_2L^{n_1+n_2}}{n_1(n_1+n_2)}\right)=1,\\
\int \mathbf{z}(\mathbf{W}^T\mathbf{D}_2^{-1}\mathbf{W})^{-1}\mathbf{z}dG^{2*}(\mathbf{z}) &= \frac{1}{\lambda_2^*}\left(\int_0^1 u_1 d\frac{\left(G_1^{2*}\right)^{n_2}}{n_2}+\int_L^1 u_2 d\frac{\left(G_2^{2*}\right)^{n_1+n_2}}{n_1+n_2}\right)\\
&= \frac{1}{\lambda_2^*}\left[\frac{u_1}{n_2}+\frac{u_2\left(1-L^{n_1+n_2}\right)}{n_1+n_2}\right]=1.
\end{aligned}
$$

Furthermore, since $\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2 \le \mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1$,

$$
\begin{aligned}
\sum_{j=1}^2 u_j G_j^{1*}(z_j)^{n_1-1}G_j^{2*}(z_j)^{n_2} &= \frac{\lambda_1^* z_1^2}{\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_1}\left[\frac{\beta_1}{\beta_2}\left(\frac{\lambda z_1^2}{u_1\beta_2}\right)^{\frac{1}{n_2-1}}\right]+\frac{\lambda_1^* z_2^2}{\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}\\
&\le \lambda_1^*\left(\frac{z_1^2}{\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_1}+\frac{z_2^2}{\mathbf{w}_2^T\mathbf{D}_1^{-1}\mathbf{w}_2}\right),\\
\sum_{j=1}^2 u_j G_j^{1*}(z_j)^{n_1}G_j^{2*}(z_j)^{n_2-1} &= \frac{\lambda_2^* z_1^2}{\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}+\frac{\lambda_2^* z_2^2}{\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2}\left[\frac{\left(\frac{\lambda_1^* z_2^2}{u_2L^{n_2}}\right)^{\frac{1}{n_1-1}}}{L}\wedge 1\right]\\
&\le \lambda_2^*\left(\frac{z_1^2}{\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}+\frac{z_2^2}{\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2}\right)
\end{aligned}
$$

for $z_1 \in \left[0, \sqrt{\frac{u_1\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1}{\lambda_1^*\mathbf{w}_2^T\mathbf{D}_2^{-1}\mathbf{w}_2}}\right]$ and $z_2 \in \left[0, \sqrt{\frac{u_2}{\lambda_1^*}}\right]$, and the equality holds at $z_1 = z_2 = 0$.

2. We can easily verify that (12) holds at $(\lambda_1^*, \lambda_2^*) = (\lambda\mathbf{w}_1^T\mathbf{D}_1^{-1}\mathbf{w}_1, \lambda\mathbf{w}_1^T\mathbf{D}_2^{-1}\mathbf{w}_1)$ following a similar argument.

3. We can also verify that G^{1*} and G^{2*} are feasible, and (12) holds at $\lambda_1^* = \frac{u_1}{n_1}$ and $\lambda_2^* = \frac{u_2}{n_2}$.

□

Proof of Proposition EC.2. The case with $n_1 = 1$ and $n_2 > 1$ and the case with $n_2 = 1$ and $n_1 > 1$ are symmetric. The proofs are almost identical, so we only present the proof with $i = 2$ and omit the other case for brevity.

1. (a) follows from an identical proof of 1(a) in Proposition EC.1. For (b), we will show that (12) holds at $(\lambda_i^*, \lambda_{i'}^*) = \left(\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i\lambda, \mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i(1-L)\lambda\right)$. Both G^{i*} and $G^{i'*}$ are feasible as

$$
\begin{aligned}
\int \mathbf{z}(\mathbf{W}^T\mathbf{D}_{i'}^{-1}\mathbf{W})^{-1}\mathbf{z}dG^{i'*}(\mathbf{z}) &= \frac{u_i}{\lambda_{i'}^*}\int_{G_i^{i*}=L^{\frac{1}{n_i}}}^1 \left[\left(G_i^{i*}\right)^{n_i}-L\right] d\left[\frac{\left(G_i^{i*}\right)^{n_i}-L}{1-L}\left(G_i^{i*}\right)^{1-n_i}\right]\\
&= \frac{u_i}{\lambda_{i'}^*}\left[\frac{-n_ih(L)}{(1-L)(n_i+1)}+1-L\right]=1,\\
\int \mathbf{z}(\mathbf{W}^T\mathbf{D}_i^{-1}\mathbf{W})^{-1}\mathbf{z}dG^{i*}(\mathbf{z}) &= \frac{1}{\lambda_i^*}\left(\int_0^1 u_{i'} d\frac{\left(G_{i'}^{i*}\right)^{n_i}}{n_i}+\int_{L^{\frac{1}{n_2}}}^1 \frac{u_i}{1-L}\left(\left(G_2^{2*}\right)^{n_2}-L\right)dG_2^{2*}\right)\\
&= \frac{1}{\lambda_i^*}\left[\frac{u_{i'}}{n_i}+\frac{u_ih(L)}{(1-L)(n_i+1)}\right]=1.
\end{aligned}
$$

Note that, when L increases from 0 to $1 - \frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}$, the left hand side of (EC.1) decreases from $\frac{u_i}{n_i+1} - \frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_in_iu_i - \mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_iu_{i'}}{(\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i+\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_in_i)n_i} \ge 0$ to $\frac{u_iC\left(\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i\cdot\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}\right)^2}{(n_i+1)\left(\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}\cdot\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i\right)^2} -$

$\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i n_i u_i - \mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i u_{i'}}{\left(\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}+\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i n_i\right)n_i} \le 0$. Therefore, there is a unique $L \in \left[0, 1 - \frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\right]$ satisfying (EC.1), at which

$$\begin{aligned}
\sum_{j=1}^{2} u_j G_j^{i*}(z_j)^{n_i} =& \frac{\lambda_{i'}^* z_{i'}^2}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}\left[\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\cdot\frac{G_{i'}^{i*}(z_{i'})}{(1-L)}\right] + \frac{\lambda_1^* z_2^2}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i} + u_i L \\
\le& \lambda_{i'}^*\left(\frac{z_{i'}^2}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}} + \frac{z_i^2}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\right) + u_i L, \\
\sum_{j=1}^{2} u_j G_j^{1*}(z_j) G_j^{2*}(z_j)^{n_2-1} =& \lambda_i^*\left(\frac{z_{i'}^2}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}} + \frac{z_i^2}{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}\right).
\end{aligned}$$

2. At $(\lambda_i^*, \lambda_{i'}^*) = \lambda\left(\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}, \mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}\right)$,

$$\begin{aligned}
\sum_{j=1}^{2} u_j G_j^{i*}(z_j)^{n_i} =& \frac{\lambda_{i'}^* z_{i'}^2}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}\left(\frac{1}{L}\left(\frac{\lambda z_{i'}^2}{u_{i'}L}\right)^{\frac{1}{n_i-1}} \wedge 1\right) \\
&+ \frac{\lambda_{i'}^* z_i^2}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i} + u_i\left(1 - \frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\right) \\
\le& \lambda_{i'}^*\left(\frac{z_{i'}^2}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}} + \frac{z_i^2}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\right) + u_i\left(1 - \frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}\right), \\
\sum_{j=1}^{2} u_j G_j^{i'*}(z_j) G_j^{i*}(z_j)^{n_i-1} =& \lambda_i^*\left(\frac{z_{i'}^2}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}} + \frac{z_i^2}{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}\right),
\end{aligned}$$

and both G^{i*} and $G^{i'*}$ are feasible as

$$\begin{aligned}
\int \mathbf{z}(\mathbf{W}^T\mathbf{D}_{i'}^{-1}\mathbf{W})^{-1}\mathbf{z}dG^{i'*}(\mathbf{z}) =& \frac{u_i}{\lambda_{i'}^*}\left[\frac{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}\cdot\frac{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}} - \frac{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}\cdot\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}\cdot\frac{n_2 C}{(n_2+1)}\right] \\
&+ \frac{u_{i'}(1-L^{n_i+1})}{\lambda_{i'}^*(n_i+1)} = 1, \\
\int \mathbf{z}(\mathbf{W}^T\mathbf{D}_i^{-1}\mathbf{W})^{-1}\mathbf{z}dG^{i*}(\mathbf{z}) =& \frac{u_{i'}}{\lambda_i^*}\frac{n_i + L^{1+n_i}}{n_i(n_i+1)} + \frac{\mathbf{w}_i^T\mathbf{D}_{i'}^{-1}\mathbf{w}_i}{\mathbf{w}_{i'}^T\mathbf{D}_{i'}^{-1}\mathbf{w}_{i'}}\cdot\frac{\mathbf{w}_{i'}^T\mathbf{D}_i^{-1}\mathbf{w}_{i'}}{\mathbf{w}_i^T\mathbf{D}_i^{-1}\mathbf{w}_i}\cdot\frac{u_i C}{\lambda_i^*(n_i+1)} = 1.
\end{aligned}$$

Thus, (12) holds. (b) holds following a similar argument.

□

Proof of Propositions 7 and EC.3. If the problem can be reduced to a single-event one with $\tilde{\mathbf{w}}$, then $\mathbf{Z}^{i*}$ is parallel to $\mathbf{W}^T\tilde{\mathbf{w}}$. As $u_i > 0$, $\tilde{\mathbf{w}} \ne \mathbf{w}_i$ for $i = 1, 2$ and $Z_2^{i*} = \hat{v}Z_1^{i*}$ for some $\hat{v} > 0$. By Theorem 1, the support of $\mathbf{Z}^{i*}$ is a subset of the maximizers of $\sum_{j=1}^{2} u_j G_j^{i*}(z_j)^{n_i-1} G_j^{i'*}(z_j)^{n_{i'}} - \lambda_i^* \mathbf{z}^T(\mathbf{W}^T\mathbf{D}_i^{-1}\mathbf{W})^{-1}\mathbf{z}$. Since $G_1^{i*}(z_1) = G_2^{i*}(\hat{v}z_1)$, the first-order conditions are

$$\frac{d_{22}^i - d_{12}^i\hat{v}}{\hat{v}u_1} = \frac{d_{11}^i\hat{v} - d_{12}^i}{u_2},\ \ i = 1, 2.$$

Eliminating $\hat{v}$ in the preceding equations yields (13) and solving the equations yields $\hat{v} = v$. The second-order condition then implies that $d_{12}^i \ge 0$, $i = 1, 2$.

At this point, we verify that the distributions in the proposition are indeed equilibria when the conditions hold.

1. We will show that (12) holds at $\lambda_i^* = \frac{\Gamma_{i'}(u_1+u_2)}{n_1\Gamma_2+n_2\Gamma_1}$, $i, i'=1,2$ and $i \neq i'$. When $L>0$, both G^{1*} and G^{2*} are feasible as

$$\int \mathbf{z}(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}dG^{1*}(\mathbf{z})$$
$$=\frac{\Gamma_1(n_1\Gamma_2+n_2\Gamma_1)}{\Gamma_1\Gamma_2}\left((n_1L)^{\frac{1}{n_1+n_2}}\int_0^{(n_1L)^{\frac{n_2}{n_1+n_2}}} d\frac{(G^{1*})^{n_1}}{n_1}+\int_{(n_1L)^{\frac{1}{n_1+n_2}}}^1 d\frac{(G^{1*})^{n_1+n_2}}{n_1+n_2}\right)=1,$$
$$\int \mathbf{z}(\mathbf{W}^T\mathbf{D}_2^{-1}\mathbf{W})^{-1}\mathbf{z}dG^{2*}(\mathbf{z})=\Gamma_2\frac{n_1\Gamma_2+n_2\Gamma_1}{\Gamma_1\Gamma_2}\int_{(n_1L)^{\frac{1}{n_1+n_2}}}^1 d\frac{(G^{2*})^{n_1+n_2}}{n_1+n_2}=1,$$

and

$$\sum_{j=1}^2 u_jG_j^{i*}(z_j)^{n_i-1}G_j^{i'*}(z_j)^{n_{i'}}-\lambda_i^*\mathbf{z}(\mathbf{W}^T\mathbf{D}_i^{-1}\mathbf{W})^{-1}\mathbf{z}\le\frac{-\lambda_i^*vd_{12}^i}{|\mathbf{W}^T\mathbf{D}_i^{-1}\mathbf{W}|}\left(z_1-\frac{z_2}{v}\right)^2\le 0$$

for $z_1\in\left[0,\sqrt{\frac{n_1\Gamma_2+n_2\Gamma_1}{\Gamma_1\Gamma_2}}\right]$ and $z_2\in\left[0,v\sqrt{\frac{n_1\Gamma_2+n_2\Gamma_1}{\Gamma_1\Gamma_2}}\right]$. When $L<0$, the statement follows from a similar argument.

2. We can directly verify that G^{i*} is feasible and (12) holds at $\lambda_1^*=\lambda_2^*=\frac{1}{2}\left(\frac{\Gamma_2}{\Gamma_1}\wedge\frac{\Gamma_1}{\Gamma_2}\right)$.

3. (a) As the left hand side of (EC.3) increases from $\frac{\Gamma_{i'}-\Gamma_i}{(n_i+1)(\Gamma_i+n_i\Gamma_{i'})}<0$ to $\frac{1}{2n_i}>0$ as L increases from 0 to 1, there is a unique $L\in[0,1]$ satisfying (EC.3). Following a similar argument as in 1, We can easily verify that G^{i*} and $G^{i'*}$ are feasible and (12) holds at $\lambda_{i'}^*=\frac{(1-L)^2\Gamma_i(u_1+u_2)}{(1-L)\Gamma_i+n_i\Gamma_{i'}}$ and $\lambda_i^*=\frac{(1-L)\Gamma_1(u_1+u_2)}{(1-L)\Gamma_i+n_i\Gamma_{i'}}$.

(b) Following a similar argument, we can easily verify that G^{i*} and $G^{i'*}$ are feasible and (12) holds at $\lambda_i^*=\frac{\Gamma_{i'}(u_1+u_2)}{\Gamma_i+n_i\Gamma_{i'}}$ and $\lambda_{i'}^*=\frac{\Gamma_i(u_1+u_2)}{\Gamma_i+n_i\Gamma_{i'}}$.

□

Proof of Proposition 8. Since $(\tilde{\mathbf{w}}_1^{(k)},\dots,\tilde{\mathbf{w}}_J^{(k)})$ are linearly independent, following a similar argument as in the proof of Theorem 1, there exists $(\lambda_{k1}^*,\dots,\lambda_{kn}^*)>\mathbf{0}$ such that

$$\sum_{i\in\mathcal{I}}\lambda_{ki}^*+\max_{\mathbf{x}\in\mathbb{R}^m,\mathbf{y}\in\mathbb{R}^J}\left\{\sum_{j\in\mathcal{J}}u_j\prod_{\ell\in\mathcal{I}_{-i}}G_j^{k\ell}\left(\mathbf{w}_j^T\mathbf{x}+\frac{1}{k}y_j\right)-\lambda_{ki}^*\left(\mathbf{x}^T\mathbf{D}_i\mathbf{x}+k\mathbf{y}^T\mathbf{y}\right)\right\}=\sum_{j\in\mathcal{J}}u_j.$$

There exists $(\mathbf{x}^{ki*},\mathbf{y}^{ki*})$ such that $(\mathbf{x}^{ki*})^T\mathbf{D}_i\mathbf{x}^{ki*}+k(\mathbf{y}^{ki*})^T\mathbf{y}^{ki*}\le 1$ and is in the support of competitor i's equilibrium distribution, which is a subset of the maximizers of the optimization problem in the above equation.

Since $\lambda_{ki}^*\in\left[0,\sum_{j\in\mathcal{J}}u_j\right]$, $\max_{\mathbf{x}\in\mathbb{R}^m,\mathbf{y}\in\mathbb{R}^J}\left\{\sum_{j\in\mathcal{J}}u_j\prod_{\ell\in\mathcal{I}_{-i}}G_j^{k\ell}\left(\mathbf{w}_j^T\mathbf{x}+\frac{1}{k}y_j\right)-\lambda_{ki}^*\left(\mathbf{x}^T\mathbf{D}_i\mathbf{x}+k\mathbf{y}^T\mathbf{y}\right)\right\}\in\left[0,\sum_{j\in\mathcal{J}}u_j\right]$, G_j^{ki} is increasing and bounded, $\mathbf{x}^{ki*}$ is bounded, and $\left\|\mathbf{y}^{ki*}\right\|_2\le\frac{1}{k}$, there exists a subsequence $\{k_r: r=1,2,\dots\}\subseteq\mathbb{N}$ and $(\lambda_i^*,G^{i*},\mathbf{x}^{i*})$, $i=1,\dots,n$, such that, as $r\to\infty$, $\lambda_{k_ri}^*\to\lambda_i^*$, $G_j^{k_ri}\to G_j^{i*}$, $j\in\mathcal{J}$, $\mathbf{x}^{k_ri*}\to\mathbf{x}^{i*}$, $\mathbf{y}^{k_ri*}\to\mathbf{0}$, and

$$\sum_{j\in\mathcal{J}}u_j\prod_{\ell\in\mathcal{I}_{-i}}G_j^{k_r\ell}\left(\mathbf{w}_j^T\mathbf{x}^{k_ri*}+\frac{1}{k_r}y_j^{k_ri*}\right)-\lambda_{k_ri}^*\left(\mathbf{x}^{k_ri*T}\mathbf{D}_i\mathbf{x}^{k_ri*}+k_r\mathbf{y}^{k_ri*T}\mathbf{y}^{k_ri*}\right)$$
$$=\max_{\mathbf{x}\in\mathbb{R}^m,\mathbf{y}\in\mathbb{R}^J}\left\{\sum_{j\in\mathcal{J}}u_j\prod_{\ell\in\mathcal{I}_{-i}}G_j^{k_r\ell*}\left(\mathbf{w}_j^T\mathbf{x}+\frac{1}{k_r}y_j\right)-\lambda_{k_ri}^*\left(\mathbf{x}^T\mathbf{D}_i\mathbf{x}+k_r\mathbf{y}^T\mathbf{y}\right)\right\}$$

converges. Furthermore, for any $\mathbf{x}\in\mathbb{R}^n$,

$$\begin{aligned}
&\sum_{j\in\mathcal{J}} u_j \prod_{\ell\in\mathcal{I}_{-i}} G_j^{\ell *}\left(\mathbf{w}_j^T\mathbf{x}^{i*}\right) - \lambda_i^*\left({\mathbf{x}^{i*}}^T\mathbf{D}_i\mathbf{x}^{i*}\right)\\
&=\lim_{r\to\infty}\sum_{j\in\mathcal{J}} u_j \prod_{\ell\in\mathcal{I}_{-i}} G_j^{k_r\ell}\left(\mathbf{w}_j^T\mathbf{x}^{k_r i*}+\frac{1}{k_r}y_j^{k_r i*}\right) - \lambda_{k_r i}^*\left({\mathbf{x}^{k_r i*}}^T\mathbf{D}_i\mathbf{x}^{k_r i*}+k_r{\mathbf{y}^{k_r i*}}^T\mathbf{y}^{k_r i*}\right)\\
&\geq \lim_{r\to\infty}\sum_{j\in\mathcal{J}} u_j \prod_{\ell\in\mathcal{I}_{-i}} G_j^{k_r\ell}\left(\mathbf{w}_j^T\mathbf{x}\right) - \lambda_{k_r i}^*\left(\mathbf{x}^T\mathbf{D}_i\mathbf{x}\right)\\
&=\sum_{j\in\mathcal{J}} u_j \prod_{\ell\in\mathcal{I}_{-i}} G_j^{\ell *}\left(\mathbf{w}_j^T\mathbf{x}\right) - \lambda_i^*\left(\mathbf{x}^T\mathbf{D}_i\mathbf{x}\right),
\end{aligned}$$

i.e., $\mathbf{x}^{i*}$ maximizes $\sum_{j\in\mathcal{J}} u_j \prod_{\ell\in\mathcal{I}_{-i}} G_j^{\ell *}\left(\mathbf{w}_j^T\mathbf{x}\right) - \lambda_i^*\left(\mathbf{x}^T\mathbf{D}_i\mathbf{x}\right)$, and hence,

$$\sum_{i\in\mathcal{I}}\lambda_i^* + \max_{\mathbf{x}\in\mathbb{R}^m,\mathbf{y}\in\mathbb{R}^J}\left\{\sum_{j\in\mathcal{J}} u_j \prod_{\ell\in\mathcal{I}_{-i}} G_j^{\ell *}(\mathbf{w}_j^T\mathbf{x}) - \lambda_i^*\left(\mathbf{x}^T\mathbf{D}_i\mathbf{x}\right)\right\} = \sum_{j\in\mathcal{J}} u_j.$$

The proposition then follows from a similar argument as in the proof of Theorem 1.

□

Proof of Proposision 9. We assume, without loss of generality, that $\mathbf{D}=\mathbf{I}$.

1. When $\mathbf{w}_j^T\mathbf{w}_{j'}\geq 0$ for all j and j', let $\Phi(\mathbf{y})=\text{diag}(\mathbf{y})\mathbf{W}^T\mathbf{W}\mathbf{y}$ be a function from $\mathbb{R}^J$ to $\mathbb{R}^J$. It is obvious that $\Phi(\mathbf{y})$ is continuous and closed. If $\mathbb{R}_+^J\not\subseteq\Phi(\mathbb{R}_+^J)$, then there exists a $\mathbf{v}>0$ in the boundary of $\Phi(\mathbb{R}_+^J)$. As Φ is closed, there exists $\mathbf{y}>0$ such that $\Phi(\mathbf{y})=\mathbf{v}$ and the Jacobian at $\mathbf{y}$,

$$\text{diag}(\mathbf{y})\mathbf{W}^T\mathbf{W}+\text{diag}\left(\sum_{j=1}^J\mathbf{w}_1^T\mathbf{w}_j y_j,\sum_{j=1}^J\mathbf{w}_2^T\mathbf{w}_j y_j,\cdots\sum_{j=1}^J\mathbf{w}_J^T\mathbf{w}_j y_j\right)$$

is positive definite. Thus, $\mathbf{v}$ cannot be on the boundary and $\mathbb{R}_+^J\subseteq\Phi(\mathbb{R}_+^J)$ must hold. As a result, there exists $\hat{\mathbf{y}}\in\mathbb{R}_+^J$ such that $\Phi(\hat{\mathbf{y}})=\frac{1}{\lambda^*}(u_1,u_2,\ldots,u_J)^T\in\mathbb{R}_+^J$ where $\lambda^*=\frac{1}{n}\left(\sum_{j=1}^J u_j\right)$.

Let $\mathbf{Z}^*=\hat{\mathbf{z}}U^{\frac{n-1}{2}}$ with distribution G^*, where U follows a uniform distribution on $[0,1]$ and $\hat{\mathbf{z}}=\frac{1}{\lambda^*}(\frac{u_1}{\hat{y}_1},\frac{u_2}{\hat{y}_2},\ldots,\frac{u_J}{\hat{y}_J})$. Since $E\left(\mathbf{Z}^{*T}\left(\mathbf{W}^T\mathbf{W}\right)^{-1}\mathbf{Z}^*\right)=\hat{\mathbf{z}}^T(\mathbf{W}^T\mathbf{W})^{-1}\hat{\mathbf{z}}E\left(U^{n-1}\right)=1$, G^* is feasible to (6) and

$$\begin{aligned}
&\lambda^* + \sum_{j=1}^J u_j {G_j^*}^{n-1}(z_j) - \lambda^*\mathbf{z}^T(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}\\
&=\ \lambda^* + \sum_{j=1}^J\frac{u_j}{\hat{z}_j^2}\left[z_j^2-(z_j\wedge 0)^2+\hat{z}_j^2-(\hat{z}_j\vee z_j)^2\right]-\lambda^*\mathbf{z}^T(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}\\
&\leq\ \frac{1}{n}\sum_{j=1}^J u_j+\lambda^*\mathbf{z}^T\left[\text{diag}\left((\mathbf{W}^T\mathbf{W})^{-1}\hat{\mathbf{z}}\right)\text{diag}(\hat{\mathbf{z}})^{-1}-\left(\mathbf{W}^T\mathbf{W}\right)^{-1}\right]\mathbf{z}.
\end{aligned}$$

Since

$$\begin{aligned}
&\mathbf{x}^T\left[\text{diag}\left(\left(\mathbf{W}^T\mathbf{W}\right)^{-1}\hat{\mathbf{z}}\right)\left(\mathbf{W}^T\mathbf{W}\right)\text{diag}\left(\left(\mathbf{W}^T\mathbf{W}\right)^{-1}\mathbf{z}\right)-\text{diag}\left(\left(\mathbf{W}^T\mathbf{W}\right)^{-1}\hat{\mathbf{z}}\right)\text{diag}\left(\hat{\mathbf{z}}\right)\right]\mathbf{x}\\
&=\sum_{j,j'}\frac{u_j}{\hat{z}_j\lambda^*}\mathbf{w}_j^T\mathbf{w}_{j'}\frac{u_{j'}}{\hat{z}_{j'}\lambda^*}x_j x_{j'}-\frac{u_j}{\hat{z}_j\lambda^*}\mathbf{w}_j^T\mathbf{w}_{j'}\frac{u_{j'}}{\hat{z}_{j'}\lambda^*}x_i^2=-\sum_{i<j}\frac{u_j u_{j'}}{\hat{z}_j\hat{z}_{j'}\lambda^{*2}}\mathbf{w}_j^T\mathbf{w}_{j'}(x_i-x_j)^2\leq 0,
\end{aligned}$$

$\text{diag}\left((\mathbf{W}^T\mathbf{W})^{-1}\hat{\mathbf{z}}\right)(\mathbf{W}^T\mathbf{W})\,\text{diag}\left((\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z}\right) - \text{diag}\left((\mathbf{W}^T\mathbf{W})^{-1}\hat{\mathbf{z}}\right)\text{diag}\,(\hat{\mathbf{z}})$ is negative semi-definite, and so is $\text{diag}\,((\mathbf{W}^T\mathbf{W})^{-1}\hat{\mathbf{z}})\,\text{diag}(\hat{\mathbf{z}})^{-1} - (\mathbf{W}^T\mathbf{W})^{-1}$. Therefore, condition (6) holds and the problem can be treated as a single-event problem with $\tilde{\mathbf{w}} = \mathbf{W}(\mathbf{W}^T\mathbf{W})^{-1}\hat{\mathbf{z}}$.

2. When $\mathbf{w}_j^T\mathbf{w}_{j'} \le 0$ for all j and j', letting $\hat{\lambda} = \frac{J^n+(n-1)(J-1)^n-nJ(J-1)^{n-1}}{nJ^n}\sum_{j=1}^J u_j$ and $\hat{\mathbf{X}}$ be a random variable with the distribution $P(\hat{\mathbf{X}} \in \{t\mathbf{w}_j | t \in [0, z_j]\}) = 1 \wedge \left[\left(\frac{J-1}{J}\right)^{n-1} + \frac{\hat{\lambda}}{u_j}z_j^2\right]^{\frac{1}{n-1}} - \frac{J-1}{J}$ for any $z_j > 0$, we have

$$E\left(\hat{\mathbf{X}}^T\hat{\mathbf{X}}\right) = \sum_{j=1}^J \int_0^{\sqrt{\frac{u_j}{\hat{\lambda}}\left[1-\left(\frac{J-1}{J}\right)^{n-1}\right]}} z_j^2 d\left[\left(\frac{J-1}{J}\right)^{n-1} + \frac{\hat{\lambda}}{u_j}z_j^2\right]^{\frac{1}{n-1}} = 1$$

and the distribution of $\hat{\mathbf{X}}$ is feasible. Since $\mathbf{w}_j^T\mathbf{w}_{j'} < 0$ for all $j \neq j'$, $\mathbf{w}_j^T\hat{\mathbf{X}} > 0$ if and only if $\hat{\mathbf{X}} \in \{t\mathbf{w}_j | t \ge 0\}$. Thus, $\hat{G}_j(z_j) := P(\mathbf{w}_j^T\hat{\mathbf{X}} \le z_j) \le 1 \wedge \left[\left(\frac{J-1}{J}\right)^{n-1} + \frac{\hat{\lambda}}{u_j}(0 \vee z_j)^2\right]^{\frac{1}{n-1}}$ and

$$\hat{\lambda} + \sum_{j=1}^J u_j \hat{G}_j^{n-1}(z_j) - \hat{\lambda}\mathbf{z}^T(\mathbf{W}^T\mathbf{W})^{-1}\mathbf{z} \le \left[\frac{1}{n} + \left(\frac{J-1}{J}\right)^n\right]\sum_{j=1}^J u_j.$$

Therefore, $\hat{G}$ is an ε-equilibrium for $\varepsilon \ge \left(\frac{J-1}{J}\right)^n \sum_{j=1}^J u_j$.

□