Fluid Models of Many-server Queues with Abandonment

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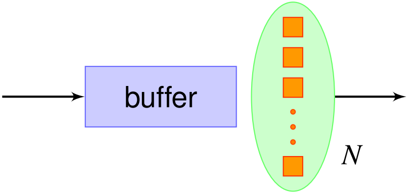
Model and Motivation

Many-server queue



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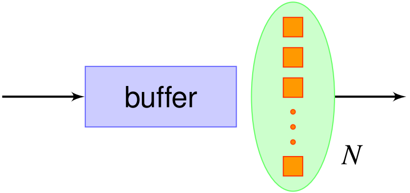

Motivation:

- Customer call centers and other services areas.

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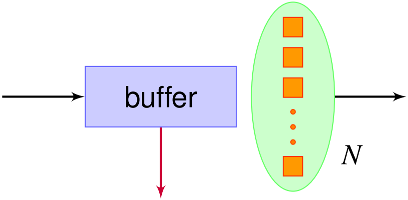
Many-server queue with abandonment



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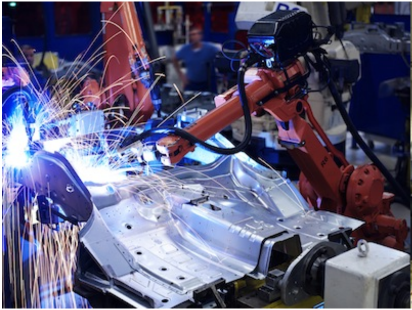
Many-server queue *v.s.* Single-server queue



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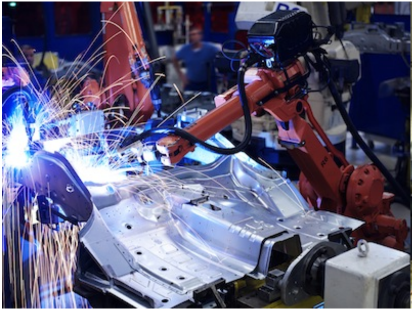


Large scale: *high demand*, need for *high capacity*

- Single server queue: increase speed
- Many-server queue: increase number of servers

A Real World Challenge

The service time is not exponentially distributed!

- Brown et. al. *Statistical analysis of a telephone call center: a queueing-science perspective*. JASA 2005

In this research

- Arrival process: general
- Service/patient time distribution: general

Literature Review

Many-server Queues

- Halfin and Whitt 1981 ($M/M/N$)
- Puhalskii and Reiman 2000 ($G/Ph/N$)
- Jelenković, Mandelbaum and Momčilović 2004 ($G/D/N$)
- Whitt 2005 ($G/H_2^*/n/m$)
- Garmarnik and Momčilović 2007 ($G/La/N$)
- Reed 2007, Puhalskii and Reed 2008 ($G/G/N$)
- Mandelbaum and Momčilović 2008 ($G/G/N$)
- Kaspi and Ramanan 2009, Kaspi 2009 ($G/G/N$)
- ......

Literature Review

Many-server Queues with Abandonment

- Whitt 2004 ($M/M/N+M$)
- Zeltyn and Mandelbaum 2005 ($G/M/N+G$)
- Whitt 2006 ($G/G/N+G$)
- Puhalskii 2008 ($M_t/M_t/N_t+M_t$)
- Kang and Ramanan 2008 ($G/G/N+G$)
- Mandelbaum and Momčilović 2009 ($G/G/N+G$)
- Dai, He and Tezcan 2009 ($G/Ph/N+G$)
- ......

System Dynamics – example with $N = 2$

$t = 0$

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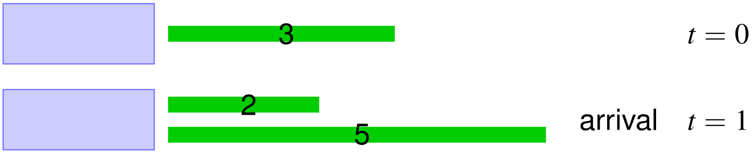

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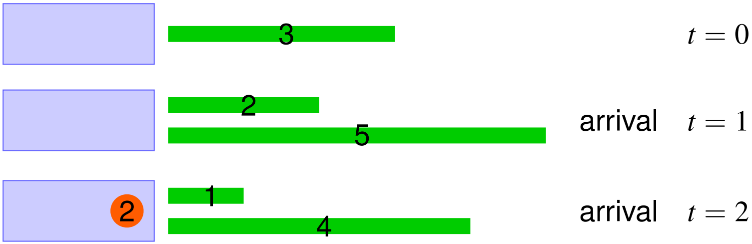

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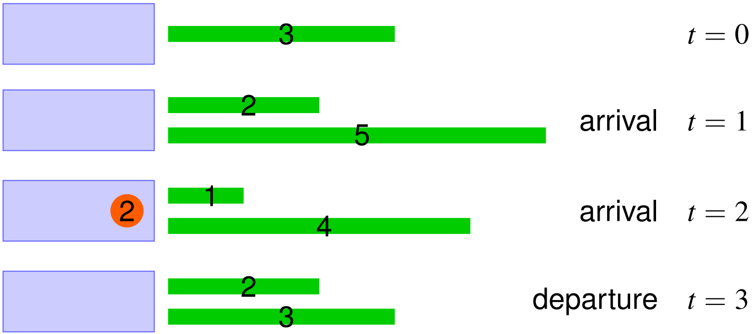

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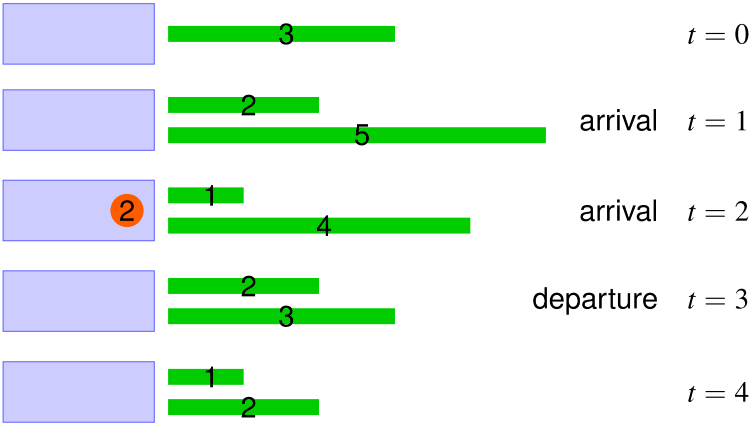

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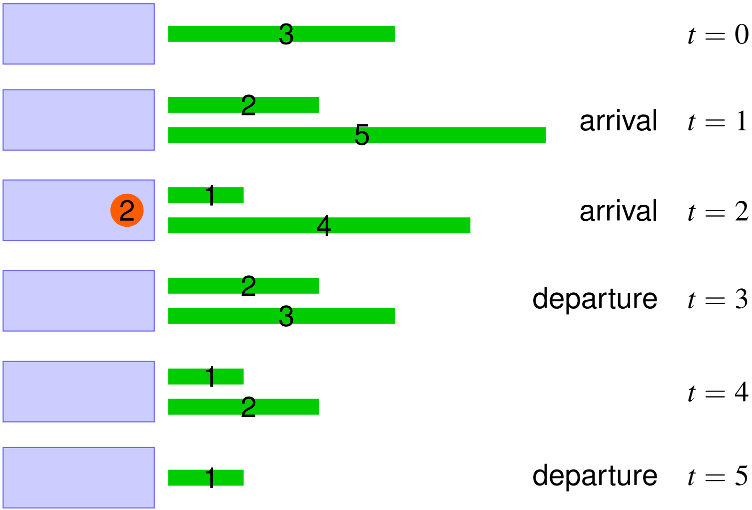

Measure-valued State Descriptor

Server pool

- $\mathcal{Z}(t)(C)$: # of customers in server with *remaining service time* in $C \subset (0, \infty)$

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Literature

Gromoll, Puha & Williams '02, Puha &Williams '02, Gromoll '06

Gromoll & Kurk '07, Gromoll, Robert & Zwart '08, ...

Zhang, Dai & Zwart '07, '08, Zhang & Zwart '08

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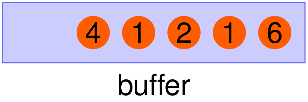

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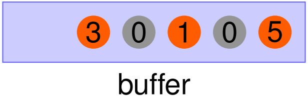

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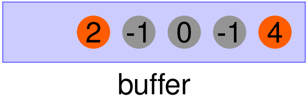

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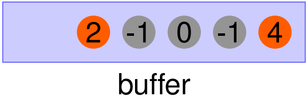

Richness

$$Q(t) = \mathcal{R}(t)((0,\infty))$$
$$R(t) = \mathcal{R}(t)((-\infty,\infty))$$

System Dynamics

Internal transfer process

$$B(t) = E(t) - R(t)$$

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Stochastic dynamic equations

$$\mathcal{R}(t)(C) = \sum_{i=1+B(t)}^{E(t)} \delta_{u_i}(C + t - a_i), \quad C \in \mathscr{B}(\mathbb{R})$$

$$\mathcal{Z}(t)(C) = \mathcal{Z}(0)(C + t) + \sum_{i=1+B(0)}^{B(t)} 1_{\{u_i > \tau_i - a_i\}} \delta_{v_i}(C + t - \tau_i), \quad C \in \mathscr{B}(\mathbb{R}_+)$$

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System Dynamics

Total number of customers

$$X(t) = Q(t) + Z(t)$$

Policy constraints

$$Q(t) = (X(t) - N)^+, \quad Z(t) = (X(t) \wedge N)$$

Fluid Model

- $E(\cdot)$: $\lambda\cdot$
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Constraints

$$\bar{Q}(t) = (\bar{X}(t) - N)^+, \quad \bar{Z}(t) = (\bar{X}(t) \wedge N)$$

Existence and Uniqueness of Fluid Model Solution

Fluid model solution with initial condition $(\bar{\mathcal{R}}_0, \bar{\mathcal{Z}}_0)$

- $(\bar{\mathcal{R}}(0), \bar{\mathcal{Z}}(0)) = (\bar{\mathcal{R}}_0, \bar{\mathcal{Z}}_0)$
- $(\bar{\mathcal{R}}(\cdot), \bar{\mathcal{Z}}(\cdot))$ satisfies fluid dynamic equations and constraints

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THEOREM

Assume that

$$G \text{ is continuous, with } 0 < \mu < \infty,$$

$$F(\cdot) \text{ is Liptachitz continuous, or } \sup_{x \in [0,\infty)} h_F(x) < \infty.$$

There exists a unique solution to the fluid model (λ, F, G, N) for any valid initial condition $(\bar{\mathcal{R}}_0, \bar{\mathcal{Z}}_0)$.

Invariant State of Fluid Model

Invariant state $(\bar{\mathcal{R}}_\infty, \bar{\mathcal{Z}}_\infty)$

- $(\bar{\mathcal{R}}(0), \bar{\mathcal{Z}}(0)) = (\bar{\mathcal{R}}_\infty, \bar{\mathcal{Z}}_\infty)$ implies $(\bar{\mathcal{R}}(\cdot), \bar{\mathcal{Z}}(\cdot)) \equiv (\bar{\mathcal{R}}_\infty, \bar{\mathcal{Z}}_\infty)$

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THEOREM

The state $(\bar{\mathcal{R}}_\infty, \bar{\mathcal{Z}}_\infty)$ is an invariant state if and only if it satisfies

$$\bar{\mathcal{R}}_\infty(C_x) = \lambda \int_0^w F^c(x+s)ds, \quad x \in \mathbb{R},$$

$$\bar{\mathcal{Z}}_\infty(C_x) = \min(\rho, 1) N[1 - G_e(x)], \quad x \in \mathbb{R}_+,$$

where w is a solution to the equation

$$F(w) = \max\left(\frac{\rho - 1}{\rho}, 0\right).$$

Fluid Model Analysis

Replace C by $C_x = (x, \infty)$, (note that $\vartheta_F(C_x) = F^c(x)$)

$$\bar{\mathcal{R}}(t)(C_x) = \lambda \int_{t-\frac{\bar{R}(t)}{\lambda}}^{t} F^c(x+t-s)ds, \quad x \in \mathbb{R},$$

$$\bar{\mathcal{Z}}(t)(C_x) = \bar{\mathcal{Z}}(0)(C_x+t) + \int_0^t F^c(\frac{\bar{R}(s)}{\lambda})G^c(x+t-s)d\bar{B}(s), \quad x \in \mathbb{R}_+,$$

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The functional equation

$$\bar{X}(t) = \zeta_0(t) + \rho \int_0^t H((\bar{X}(t-s)-1)^+)dG_e(s) + \int_0^t (\bar{X}(t-s)-1)^+ dG(s)$$

where $H(x) = F^c(F_e^{-1}(\frac{\alpha}{\lambda}x))$.

The Special Case with Exponential Distribution

Now, we specialize in the case with exponential distribution, i.e.

$$F(t) = F_e(t) = 1 - e^{-\alpha t}, \quad G(t) = G_e(t) = 1 - e^{-\mu t}.$$

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Now the key equation becomes

$$\bar{X}(t) = \zeta_0(t) + \rho \int_0^t \left[1 - \frac{\alpha}{\lambda}\left((\bar{X}(t-s) - 1)^+\right)\right] \mu e^{-\mu s} ds$$
$$+ \int_0^t (\bar{X}(t-s) - 1)^+ \mu e^{-\mu s} ds,$$

with $\zeta_0(t) = \bar{X}_0 e^{-\mu t}$.

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with $\zeta_0(t) = \bar{X}_0 e^{-\mu t}$. After some algebra, we get

$$\bar{X}'(t) = \mu(\rho - 1) - \alpha(\bar{X}(t) - 1)^+ + \mu(\bar{X}(t) - 1)^-. \quad \text{(Whitt 04)}$$

Fluid Scaling and Limiting Regimes

A sequence of systems indexed by the number of servers n.

Fluid scaling

$$\bar{\mathcal{R}}^n(t) = \frac{1}{n}\mathcal{R}^n(t), \quad \bar{\mathcal{Z}}^n(t) = \frac{1}{n}\mathcal{Z}^n(t),$$

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$$\rho^n = \frac{\lambda^n}{n\mu_n} \to \rho \quad \in (0, \infty)$$

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Constraints

$$\bar{Q}^n(t) = (\bar{X}^n(t) - 1)^+, \quad \bar{Z}^n(t) = (\bar{X}^n(t) \wedge 1)$$

Functional Law of Large Numbers

Assumption $\mathbf{A}$:

1. $\bar{E}^n(\cdot) \Rightarrow \lambda \cdot$
2. $\vartheta_F^n \to \vartheta_F$, $\vartheta_G^n \to \vartheta_G$
3. $\mu^n \to \mu$
4. $(\bar{\mathcal{R}}^n(0), \bar{\mathcal{Z}}^n(0)) \Rightarrow (\bar{\mathcal{R}}_0, \bar{\mathcal{Z}}_0)$
5. $\bar{\mathcal{R}}_0$ and $\bar{\mathcal{Z}}_0$ has no atoms

THEOREM

Under assumption $\mathbf{A}$

$$(\bar{\mathcal{R}}^n(\cdot), \bar{\mathcal{Z}}^n(\cdot)) \Rightarrow (\bar{\mathcal{R}}(\cdot), \bar{\mathcal{Z}}(\cdot)) \quad as\ n \to \infty,$$

where $(\bar{\mathcal{R}}(\cdot), \bar{\mathcal{Z}}(\cdot))$ *is almost surely the fluid model solution to* $(\lambda, F, G, 1)$ *with initial condition* $(\bar{\mathcal{R}}_0, \bar{\mathcal{Z}}_0)$.

Performance Evaluation

Approximation formulas

- $\mathbb{E}(W|S) = w, \quad F(w) = \max\left((\rho - 1)/\rho, 0\right)$
- $\mathbb{E}(Q) = \dfrac{\lambda}{\alpha} F_e(w)$

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$M/GI/100\text{-}GI$, $\lambda = 120$, $\mu = 1$, $\alpha = 1$ (Whitt 2006)

Abd.	Ser.	$\mathbb{E}[Q]$	$\mathbb{E}[W\|S]$
E_2	E_2	40.25 ± 0.057	0.353 ± 0.00051
	LN$(1,4)$	39.56 ± 0.097	0.343 ± 0.00094
Approximation		41.11	0.365
LN$(1,4)$	E_2	14.51 ± 0.018	0.126 ± 0.00017
	LN$(1,4)$	14.52 ± 0.043	0.125 ± 0.00027
Approximation		14.63	0.131

A Missing Gap

Interchange of *Steady State* and *Heavy Traffic* Limits

$$(\bar{\mathcal{R}}^n(t), \bar{\mathcal{Z}}^n(t)) \xrightarrow{\quad t \to \infty \quad} (\bar{\mathcal{R}}^n_\infty, \bar{\mathcal{Z}}^n_\infty)$$

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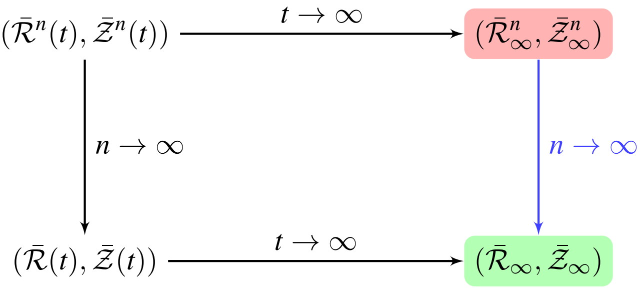
A Missing Gap

Interchange of *Steady State* and *Heavy Traffic* Limits



Questions?

Thank you!