Electronic Companion of "Online Demand Fulfillment under Limited Flexibility"

EC.1. Proof of Main Results

Proof of Theorem 1. As discussed in §5.1, the key is to to find an appropriate lower bound of Γ (defined in (7)) involving the potential function $\Phi(\mathbf{X}(k))$. We divide the proof in the following three steps.

Network Partition Let $X_{\min}(k)$ be the value of the smallest load deviation among all resources after kth arrival and $\mathcal{S}^c = \{i \in \mathcal{I} : X_i(k) = X_{\min}(k)\}$ be the set of resources with the smallest load deviation. $\mathcal{I}$ is then divided into two sets, $\mathcal{S}^c$ and $\mathcal{S} = \mathcal{I} \setminus \mathcal{S}^c$, which leads to a division of $\mathcal{J}$ into $\mathcal{J}(\mathcal{S}^c)$ and $\mathcal{T} = \mathcal{J} \setminus \mathcal{J}(\mathcal{S}^c)$ as illustrated in Figure 4. We assume for now that $\mathcal{S} \neq \emptyset$ and will get back to it later. Thus, the resource and request nodes are divided into two disjoint pairs, $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{S}^c, \mathcal{J}(\mathcal{S}^c))$. Under the load deviation policy, any request in $\mathcal{T}$ will only be assigned to and fulfilled by a resource in $\mathcal{S}$, while any request in $\mathcal{J}(\mathcal{S}^c)$ will be assigned to (but not necessarily fulfilled by) a resource in $\mathcal{S}^c$.

In the subnetwork with nodes $(\mathcal{S}, \mathcal{T})$ and arcs in $\{(i,j) \in \mathscr{A}, i \in \mathcal{S}, j \in \mathcal{T}\}$, let $\eta^{\mathcal{T}} = \sum_{i\in\mathcal{S}} c_i - \sum_{j\in\mathcal{T}} p_j = -\left(\sum_{i\in\mathcal{S}^c} c_i - \sum_{j\in\mathcal{J}(\mathcal{S}^c)} p_j\right)$. Since $X_i(k) = X_{i^*(j)}(k) = X_{\min}(k)$ for all $i \in \mathcal{S}^c$ and $j \in \mathcal{J}(\mathcal{S}^c)$, the terms in Γ (defined in (7)) for $i \in \mathcal{S}^c$ and $j \in \mathcal{J}(\mathcal{S}^c)$ can be written as

$$\sum_{i\in\mathcal{S}^c} c_i e^{X_i(k)/C} - \sum_{j\in\mathcal{J}(\mathcal{S}^c)} p_j e^{X_{i^*(j)}(k)/C} = e^{X_{\min}(k)/C}\left(\sum_{i\in\mathcal{S}^c} c_i - \sum_{j\in\mathcal{J}(\mathcal{S}^c)} p_j\right) = -\eta^{\mathcal{T}} e^{X_{\min}(k)/C}$$

and $\Gamma = \Gamma_{(\mathcal{S},\mathcal{T})} - \eta^{\mathcal{T}} e^{X_{\min}(k)/C}$ where

$$\Gamma_{(\mathcal{S},\mathcal{T})} = \sum_{i\in\mathcal{S}} c_i e^{X_i(k)/C} - \sum_{j\in\mathcal{T}} p_j e^{X_{i^*(j)}(k)/C}.$$

Bounding $\Gamma_{(\mathcal{S},\mathcal{T})}$ Using the Max-Flow Min-Cut Theorem In the subnetwork with resources $\mathcal{S}$ and request types $\mathcal{T}$ as illustrated in Figure EC.1, $\sum_{i\in\mathcal{S}} c_i - \sum_{j\in\mathcal{T}} p_j = \eta^{\mathcal{T}} \geq \eta > 0$. If we take out a total of η amount of capacity, the modified network still satisfies the nonnegative GCG. Since we are trying to find a tight lower bound of $\Gamma_{(\mathcal{S},\mathcal{T})}$, the operation of "take out" should start with resource nodes with larger $X_i(k)$. Suppose that $X_1(k) \geq \cdots \geq X_{I'}(k)$, then the "take out" should start from resource 1. Evidently, there exists a resource ℓ, $\ell \in \mathcal{S}$, such that $\eta = \sum_{i=1}^{\ell-1} c_i + c^{res}$ where $0 < c^{res} \leq c_\ell$. The capacity of the subnetwork becomes

$$c_i' = \begin{cases} 0, & \text{if } i < \ell, \\ c_\ell - c^{res,} & \text{if } i = \ell, \\ c_i, & \text{if } i > \ell. \end{cases}$$

and the GCG of the subnetwork is still non-negative, i.e., $\sum_{i\in\mathcal{I}(\mathcal{J}')} c_i' \geq \sum_{j\in\mathcal{J}'} p_j$ for any non-empty $\mathcal{J}' \subset \mathcal{T}$.

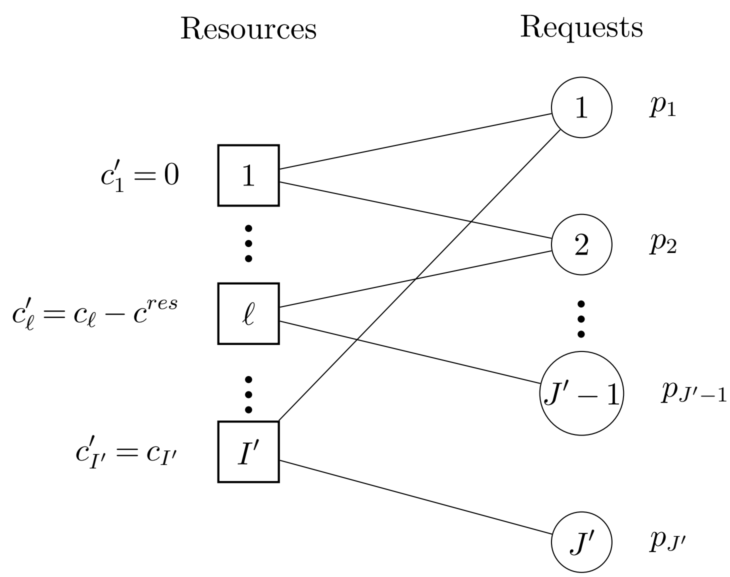

Figure EC.1 **An illustration of the subnetwork with** $(\mathcal{S},\mathcal{T})$

By the max-flow min-cut theorem, there exists a set of flows from $\mathcal{S}$ to $\mathcal{T}$, $\{f_{ij} \geq 0 : i \in \mathcal{S},\ j \in \mathcal{T}\}$ such that $f_{ij} = 0$ if $(i,j) \notin \mathscr{A}$, and

$$\sum_{j\in\mathcal{T}} f_{ij} = c_i', \text{ for all } i \in \mathcal{S}, \tag{EC.1}$$

$$\sum_{i\in\mathcal{I}(j)} f_{ij} \geq p_j, \text{ for all } j \in \mathcal{T}. \tag{EC.2}$$

It then follows that

$$\begin{aligned}
&\sum_{i\in\mathcal{S}} c_i' e^{X_i(k)/C} - \sum_{j\in\mathcal{T}} p_j e^{X_{i^*(j)}(k)/C} \\
&= \sum_{i\in\mathcal{S}} \left(\sum_{j\in\mathcal{T}} f_{ij}\right) e^{X_i(k)/C} - \sum_{i\in\mathcal{I}(j)} \sum_{j\in\mathcal{T}} f_{ij} e^{X_{i^*(j)}(k)/C} + \sum_{j\in\mathcal{T}} \left(\sum_{i\in\mathcal{I}(j)} f_{ij} - p_j\right) e^{X_{i^*(j)}(k)/C} \\
&\geq \sum_{j\in\mathcal{T}} \left(\sum_{i\in\mathcal{I}(j)} f_{ij} e^{X_i(k)/C} - \sum_{i\in\mathcal{I}(j)} f_{ij} e^{X_{i^*(j)}(k)/C}\right) + \sum_{j\in\mathcal{T}} \left(\sum_{i\in\mathcal{I}(j)} f_{ij} - p_j\right) e^{X_{\min}(k)/C} \\
&\geq \sum_{j\in\mathcal{T}} \left(\sum_{i\in\mathcal{I}(j)} f_{ij} - \sum_{i\in\mathcal{I}(j)} f_{ij}\right) e^{X_{i^*(j)}(k)/C} + \left(\sum_{i\in\mathcal{S}} c_i' - \sum_{j\in\mathcal{T}} p_j\right) e^{X_{\min}(k)/C} \\
&= (\eta^{\mathcal{T}} - \eta) e^{X_{\min}(k)/C},
\end{aligned} \tag{EC.3}$$

where the first equality and inequality follows from (EC.1), (EC.2) and the second inequality follows from the definition of $i^*(j)$.

In view of (EC.3), $\Gamma_{(\mathcal{S},\mathcal{T})}$ can be rewritten as

$$\begin{aligned}
\Gamma_{(\mathcal{S},\mathcal{T})} &= \sum_{i\in\mathcal{S}} c_i' e^{X_i(k)/C} - \sum_{j\in\mathcal{T}} p_j e^{X_{i^*(j)}(k)/C} + \sum_{i=1}^{\ell-1} c_i e^{X_i(k)/C} + c^{res} e^{X_\ell(k)/C} \\
&\geq \sum_{i=1}^{\ell-1} c_i e^{X_i(k)/C} + c^{res} e^{X_\ell(k)/C} + (\eta^{\mathcal{T}} - \eta) e^{X_{\min}(k)/C}.
\end{aligned} \tag{EC.4}$$

Recall that $\Gamma = \Gamma_{(\mathcal{S},\mathcal{T})} - \eta^{\mathcal{T}} e^{X_{\min}(k)/C}$. As a consequence of (EC.4), the following lemma establishes the relationship between Γ and the potential function $\Phi(\mathbf{X}(k))$.

Lemma EC.1. *For $\theta = \min\{\eta, I c_{\min}\}$,*

$$\Gamma \geq \frac{\theta}{I} \sum_{i=1}^{I} e^{X_i(k)/C} - \theta, \tag{EC.5}$$

and

$$\Gamma \geq \eta \sum_{i=1}^{I} c_i e^{X_i(k)/C} - \eta,$$

which is equivalent to

$$\sum_{i=1}^{I} c_i e^{X_i(k)/C} \leq \frac{\Gamma}{\eta} + 1. \tag{EC.6}$$

The Contraction Mapping By (EC.5), (EC.6) and Lemma 3, we have the contraction mapping

$$\begin{aligned}
&\mathbb{E}\big[\Phi\big(\mathbf{X}(k+1)\big)\mid \mathbf{X}(k)\big]-\Phi\big(\mathbf{X}(k)\big)\\
&\leq \frac{2}{C^2}(\frac{\Gamma}{\eta}+1)-(\frac{1}{C}+\frac{1}{C^2})\Gamma=\frac{2}{C^2}-(\frac{1}{C}+\frac{1}{C^2}-\frac{2}{C^2\eta})\Gamma\\
&\leq \frac{2}{C^2}-(\frac{1}{C}+\frac{1}{C^2}-\frac{2}{C^2\eta})\left(\frac{\theta}{I}\sum_{i=1}^{I}e^{X_i(k)/C}-\theta\right)\\
&=-a\Phi\big(\mathbf{X}(k)\big)+b
\end{aligned} \tag{EC.7}$$

for all C such that $\frac{1}{C}+\frac{1}{C^2}-\frac{2}{C^2\eta}>0$, where

$$a=\left(\frac{1}{C}+\frac{1}{C^2}-\frac{2}{C^2\eta}\right)\frac{\theta}{I},\quad b=\left(\frac{1}{C}+\frac{1}{C^2}-\frac{2}{C^2\eta}\right)\theta+\frac{2}{C^2}.$$

If $\mathcal{S}=\emptyset$, then $\mathcal{S}^c=\mathcal{I}$ and $X_i(k)=0$ for all k as $\sum_{i=1}^{I}X_i(k)=0$. Suppose that the $(k+1)$th arrival is of type j. Then, $X_i(k+1)=-c_i$ for $i\neq i^*(j)$ and $X_{i^*(j)}(k+1)=1-c_{i^*(j)}$. Hence,

$$\Phi\big(\mathbf{X}(k+1)\big)\leq I+\sum_{i=1}^{I}X_i(k+1)/C+\sum_{i=1}^{I}X_i^2(k+1)/C^2\leq I+\frac{2}{C^2}$$

where the first inequality follows from $e^x\leq 1+x+x^2$ for any $x\in[-1,1]$. Since $\Phi(\mathbf{X}(k))=I$, (EC.7) holds. Thus, (EC.7) holds for all $\mathcal{S}$ for the right choice of C.

Since $\mathbb{E}[\Phi(\mathbf{X}(0))]=I\leq I\left[1+\frac{2\eta}{(\eta C+\eta-2)\theta}\right]$, it follows that $\mathbb{E}[\Phi(\mathbf{X}(K))]\leq\frac{b}{a}=I\left[1+\frac{2\eta}{(\eta C+\eta-2)\theta}\right]$ by induction. By Lemma 2, the total number of expected number of lost sales is bounded from above by

$$\mathbb{E}\sum_{i=1}^{I}\max\{X_i(K),0\}\leq IC\ln\Big(1+1+\frac{2\eta}{(\eta C+\eta-2)\theta}\Big)\leq IC\ln\Big(2+\frac{2\eta}{(\eta C-2)\theta}\Big).$$

Theorem 1 is established at $C=\frac{3}{\theta}$ in which case $\frac{1}{C}+\frac{1}{C^2}-\frac{2}{C^2\eta}>0$ and $a<1$. □

Proof of Proposition 1. First we show that $\eta^*\leq\min_{j\in\mathcal{J}}\frac{p_j}{d(j)}$. As a matter of fact, an inventory allocation $\mathbf{c}$ that results in a positive GCG for the original system must satisfy $\sum_{i\in\mathcal{I}(\mathcal{J}')}c_i-\sum_{j\in\mathcal{J}'}p_j\geq\eta$ for any of the $d(j)$ subnetworks $\mathcal{J}'\in\mathcal{J}$. Summing up the $d(j)$ inequalities, we have $\sum_{i\in\mathcal{I}}c_i-\sum_{j'\neq j}p_{j'}=1-(1-p_j)=p_j\geq d(j)\eta$. Thus, $\eta^*\leq\min_{j\in\mathcal{J}}\frac{p_j}{d(j)}$. Combined with the fact that $d(j)=|\mathcal{I}(j)|$, $\min_{j\in\mathcal{J}}\frac{p_j}{|\mathcal{I}(j)|}$ is actually the highest GCG achievable for a connected network with $I+J-1$ arcs.

Now we discuss the case of $I+J$ arcs. For any $\mathcal{J}' \subsetneq \mathcal{J}$, $\sum_{j\in\mathcal{J}'} p_j$ amount of inventory is allocated to $\mathcal{I}(\mathcal{J}')$. Furthermore, there exists $(\hat{\imath},\hat{\jmath}) \in \mathscr{A}$ where $\hat{\jmath} \in \mathcal{J}\setminus\mathcal{J}'$ and $\hat{\imath}\in\mathcal{I}(\mathcal{J}')$. If $(\hat{\imath},\hat{\jmath})$ is unique, $\sum_{i\in\mathcal{I}(\mathcal{J}')} c_i - \sum_{j\in\mathcal{J}'} p_j \ge \frac{p_{\hat{\jmath}}}{d(\hat{\jmath})} \ge \min_{j\in\mathcal{J}} \frac{p_j}{d(j)}$. Otherwise, there exist at least two arcs between $\mathcal{I}(\mathcal{J}')$ and $\mathcal{J}\setminus\mathcal{J}'$, say from request types $\hat{\jmath}_1$ and $\hat{\jmath}_2$ (it is possible that $\hat{\jmath}_1=\hat{\jmath}_2$). With a total of $I+J$ arcs and a single cycle in the network, at least $\frac{p_{\hat{\jmath}_1}}{2d(\hat{\jmath}_1)}$ amount of $p_{\hat{\jmath}_1}$ and $\frac{p_{\hat{\jmath}_2}}{2d(\hat{\jmath}_2)}$ amount of $p_{\hat{\jmath}_2}$ are allocated to the set $\mathcal{I}(\mathcal{J}')$ in addition to $\sum_{j\in\mathcal{J}'} p_j$. Thus, $\sum_{i\in\mathcal{I}(\mathcal{J}')} c_i - \sum_{j\in\mathcal{J}'} p_j \ge \frac{p_{\hat{\jmath}_1}}{2d(\hat{\jmath}_1)} + \frac{p_{\hat{\jmath}_2}}{2d(\hat{\jmath}_2)} \ge \min_{j\in\mathcal{J}} \frac{p_j}{d(j)}$. □

Proof of Proposition 2. Let $\bar{p}_{\mathcal{J}^*} = \frac{\sum_{j\in\mathcal{J}^*} p_j}{d(\mathcal{J}^*)}$. Following a similar argument as in the proof of Proposition 1, we can easily show that $\eta^* \le \bar{p}_{J^*}$ and hence it suffices to prove that the inventory allocation described in §6.1.2 achieves the GCG of $\bar{p}_{J^*}$. Recall that $\eta = \min_{\mathcal{J}'\subset\mathcal{J}} \eta^{\mathcal{J}'}$. We will show that $\eta^{\mathcal{J}'} \ge \bar{p}_{\mathcal{J}^*}$ for any $\mathcal{J}'\subset\mathcal{J}$ by considering the following three cases.

1. If $\mathcal{J}'\cap\mathcal{J}^* = \emptyset$, the request nodes in $\mathcal{J}'$ in different subnetworks have no common suppliers and the desired conclusion holds from the definition of $\bar{p}_{\mathcal{J}^*}$.

2. If $\mathcal{J}'\cap\mathcal{J}^* = \mathcal{J}^*$, we can break $\sum_{i\in\mathcal{I}(\mathcal{J}')} c_i - \sum_{j\in\mathcal{J}'} p_j$ into $d(\mathcal{J}^*)$ terms based on their association with the subnetworks. Since $\sum_{j\in\mathcal{J}^*} p_j = d(\mathcal{J}^*)\bar{p}_{\mathcal{J}^*}$, we can establish the desired result for each subnetwork.

3. Otherwise, let $\mathcal{J}'\cap\mathcal{J}^* = \Omega$. Suppose that there are m subnetworks with at least one resource in $\mathcal{I}(\Omega)$ and m_0 subnetworks with at least one resource in $\mathcal{I}(\mathcal{J}'\setminus\Omega)$. Then, $\mathcal{J}' = \mathcal{J}'_1 \cup\cdots\cup \mathcal{J}'_{m_0}\cup\Omega$, where $\mathcal{J}'_j$ belongs to different subnetworks. under our inventory allocation $\mathbf{c}$, $\sum_{i\in\mathcal{I}(\mathcal{J}'_k)} c_i - \sum_{j\in\mathcal{J}'_k} p_j \ge \bar{p}_{\mathcal{J}^*}$ for any $k\in\{1,\cdots,m_0\}$. In addition, there are at least $m-m_0$ suppliers in $\mathcal{I}(\Omega)\setminus\mathcal{I}(\mathcal{J}'_1\cup\cdots\cup\mathcal{J}'_{m_0})$, each of which has a capacity of at least $\bar{p}_{\mathcal{J}^*}$. It follows that

$$\begin{aligned}\sum_{i\in\mathcal{I}(\mathcal{J}')} c_i - \sum_{j\in\mathcal{J}'} p_j &= \sum_{1\le k\le m_0}\left(\sum_{i\in\mathcal{I}(\mathcal{J}'_k)} c_i - \sum_{j\in\mathcal{J}'_k} p_j\right) + \sum_{i\in\mathcal{I}(\Omega)\setminus\mathcal{I}(\mathcal{J}'_1\cup\cdots\cup\mathcal{J}'_{m_0})} c_i - \sum_{j\in\Omega} p_j \\ &\ge m_0\bar{p}_{\mathcal{J}^*} + (m-m_0)\bar{p}_{\mathcal{J}^*} - \sum_{j\in\Omega} p_j = m\bar{p}_{\mathcal{J}^*} - \sum_{j\in\Omega} p_j.\end{aligned}$$

Since $d(\mathcal{J}^*) = d(\mathcal{J}^*\setminus\Omega)+m-1$, $\bar{p}_{\mathcal{J}^*} = \frac{\sum_{j\in\mathcal{J}^*} p_j}{d(\mathcal{J}^*)} \le \frac{\sum_{j\in\mathcal{J}^*\setminus\Omega} p_j}{d(\mathcal{J}^*\setminus\Omega)} = \frac{\sum_{j\in\mathcal{J}^*\setminus\Omega} p_j}{d(\mathcal{J}^*)-(m-1)}$, which implies that $\sum_{j\in\Omega} p_j \le (m-1)\bar{p}_{\mathcal{J}^*}$. Desired result thus follows. □

Proof of Proposition 3. We first note that the value of η^* is strictly increasing with each iteration and hence the algorithm ends in a finite number of steps. In each iteration, we remove one resource from a request type with the lowest ratio $\frac{p_j}{|\mathcal{I}(j)|}$, i.e., $\underline{j}$, and add one resource to the request type with the highest ratio $\frac{p_j}{|\mathcal{I}(j)|+1}$, i.e., $\bar{j}$. If the algorithm finds a solution with $|\mathcal{I}(\underline{j})|=1$, then $\eta^*=p_1=p_{\min}$ as p_j is non-decreasing in j. Otherwise, if the algorithm finds a solution such that $\frac{p_{\underline{j}}}{|\mathcal{I}(\underline{j})|} \geq \frac{p_{\bar{j}}}{|\mathcal{I}(\bar{j})|+1}$, then $\eta^* \geq \frac{p_j}{|\mathcal{I}(j)|+1}$ for all $j \in \mathcal{J}$, i.e., $|\mathcal{I}(j)|+1$ is the maximum number of resources that can supply request type j in a network with the highest GCG. Thus, any solution that differs from the current one would have one request type linked to at least $|\mathcal{I}(j)|+1$ resources and hence its GCG cannot be higher. □

Proof of Theorem 2. The proof follows that of Theorem 1 with a modification to Lemma 3 whose proof can be found in EC.2.

LEMMA EC.2. *Let* $A = \max_j \frac{\mathbb{E}(\ell_j^2)}{\mathbb{E}(\ell_j)}$. *For any* $C \geq \bar{\ell}$,

$$\mathbb{E}\big[\Phi\big(\mathbf{X}(k+1)\big) \mid \mathbf{X}(k)\big] - \Phi\big(\mathbf{X}(k)\big) \leq \frac{2AD}{C^2}\sum_{i=1}^{I} c_i e^{X_i(k)/C} - \left(\frac{D}{C}+\frac{AD}{C^2}\right)\Gamma,$$

where $\Gamma = \left(\sum_{i=1}^{I} c_i e^{X_i(k)/C} - \sum_{j=1}^{J} p'_j e^{X_{i^*(j)}(k)/C}\right)$ *and* $i^*(j)$ *is the resource assigned to the* $(k+1)$*-th arrival if it is of type* j.

By Lemma EC.2, we obtain a similar contraction mapping

$$\begin{aligned}&\mathbb{E}\big[\Phi\big(\mathbf{X}(k+1)\big) \mid \mathbf{X}(k)\big] - \Phi\big(\mathbf{X}(k)\big)\\ \leq\;& D\left[\left(\frac{1}{C}+\frac{A}{C^2}-\frac{2A}{C^2\eta'}\right)\theta' + \frac{2A}{C^2}\right] - D\left(\frac{1}{C}+\frac{A}{C^2}-\frac{2A}{C^2\eta'}\right)\frac{\theta'}{I}\sum_{i=1}^{I} e^{X_i(k)/C},\end{aligned}$$

where $\theta' = \min\{\eta', Ic_{\min}\}$. With the choice of $C = \max\{\bar{\ell}, 3A/\theta'\}$,

$$\sum_{i=1}^{I}\mathbb{E}X_i^+(\mathcal{D}) = \sum_{i=1}^{I}\mathbb{E}\left[L_i(\mathcal{D}) - c_i\sum_{k=1}^{\mathcal{D}}\ell(k)\right]^+ \leq \ln 64 \cdot \max\left\{\bar{\ell}, \frac{A}{\min\{c_{\min}, \eta'\}}\right\}$$

where $\mathcal{D}$ is the random variable representing the total number of arrivals.

Note that the expected number of lost sales under the full flexibility case is $\mathbb{E}\left[\sum_{k=1}^{\mathcal{D}}\ell(k) - K\right]^+$ for given $\mathcal{D}$, the expected optimality gap is bounded by

$$\sum_{i=1}^{I}\mathbb{E}\left[L_i(\mathcal{D}) - c_iK\right]^+ - \mathbb{E}\left[\sum_{k=1}^{\mathcal{D}}\ell(k) - K\right]^+ \leq \sum_{i=1}^{I}\mathbb{E}\left[L_i(\mathcal{D}) - c_i\sum_{k=1}^{\mathcal{D}}\ell(k)\right]^+ \leq \ln 64 \cdot \max\left\{\bar{\ell}, \frac{A}{\min\{c_{\min}, \eta'\}}\right\}$$

□

Proof of Theorem 3. Following the proof of Theorem 1, we can show the following inequality similar to (EC.7):

$$\mathbb{E}\left[\Phi\big(\mathbf{X}(k+1)\big) \mid \mathbf{X}(k)\right] - \Phi\big(\mathbf{X}(k)\big) \leq -a_k\Phi\big(\mathbf{X}(k)\big) + b_k \tag{EC.8}$$

for all C such that $\frac{1}{C}+\frac{1}{C^2}-\frac{2}{C^2\eta_k}>0$, where

$$a_k=\left(\frac{1}{C}+\frac{1}{C^2}-\frac{2}{C^2\eta_k}\right)\frac{\theta_k}{I},\quad b_k=\left(\frac{1}{C}+\frac{1}{C^2}-\frac{2}{C^2\eta_k}\right)\theta_k+\frac{2}{C^2},\quad \theta_k=\min\{\eta_k, Ic_{\min}\}.$$

We first present a lemma that will be used later.

Lemma EC.3. *Suppose that a sequence $\{e_k\}$ satisfies $e_{k+1}\leq(1-a_k)e_k+b$ for all k and $e_1\leq b$, where $a_k<1, b>0$. Then, for $K>0$,*

$$e_K\leq\frac{b}{\min\limits_k\left\{\frac{1}{k}\sum\limits_{r=K-k}^{K-1}a_r\right\}}.$$

Rewrite (EC.8) as

$$\mathbb{E}\left[\Phi\big(\mathbf{X}(k+1)\big)-I \mid \mathbf{X}(k)\right]\leq(1-a_k)(\Phi\big(\mathbf{X}(k)\big)-I)+\frac{2}{C^2}.$$

Applying Lemma EC.3 with $e_k=\mathbb{E}[\Phi\big(\mathbf{X}(k)\big)]-I$, we have

$$\mathbb{E}[\Phi\big(\mathbf{X}(k)]\leq I+\frac{2/C^2}{\min\limits_k\left\{\frac{1}{k}\sum\limits_{r=K-k+1}^{K}a_r\right\}}.$$

Letting $C=\frac{3}{\min\limits_{1\leq k\leq K}\{\bar{\theta}_k\}}$, we can verify that, for all $0<k\leq K$, $\frac{1}{k}\sum\limits_{r=K-k+1}^{K}a_r\geq 1/(IC^2)$. Hence, $\mathbb{E}[\Phi\big(\mathbf{X}(k)]\leq 3I$ and

$$\mathbb{E}\sum_{i=1}^{I}\max\left\{X_i(K),0\right\}\leq IC\ln(4)=\frac{I\ln 64}{\min\limits_{1\leq k\leq K}\{\bar{\theta}_k\}}.$$

□

EC.2. Proof of lemmas

Proof of Lemma 1. Suppose that there exists a $\mathcal{J}'\subset\mathcal{J}$ such that $\sum\limits_{i\in\mathcal{I}(\mathcal{J}')}c_i\leq\sum\limits_{j\in\mathcal{J}'}p_j<1$. Since the total number of requests from $\mathcal{J}'$ follows a Binomial distribution with

$\left(K, \sum_{j\in\mathcal{J}'} p_j\right)$ and there is only $\left(\sum_{i\in\mathcal{I}(\mathcal{J}')} c_i\right) K$ amount of supply, the total expected number of lost sales are at least

$$
\begin{aligned}
&\mathbb{E}\left[\text{Binom}\left(K, \sum_{j\in\mathcal{J}'} p_j\right) - \left(\sum_{i\in\mathcal{I}(\mathcal{J}')} c_i\right) K\right]^+ \\
\approx\ &\mathbb{E}\left[\left(\sum_{j\in\mathcal{J}'} p_j - \sum_{i\in\mathcal{I}(\mathcal{J}')} c_i\right) K + \sqrt{K}\text{Norm}\left(0, \left(\sum_{j\in\mathcal{J}'} p_j\right)\left(1 - \sum_{j\in\mathcal{J}'} p_j\right)\right)\right]^+
\end{aligned}
$$

by the Central Limit Theorem, which is at least in the order of $\sqrt{K}$. □

Proof of Lemma 2. Consider a variant of the LDP in which any request assigned to a resource after it runs out of inventory is simply lost, instead of (possibly) fulfilled by another resource. We claim that the expected total number of lost sales is exactly $\mathbb{E}\sum_{i=1}^{I}\max\{X_i(K), 0\}$ under the new policy. Indeed, a request assigned to resource i is lost if and only if $L_i(k) > c_i K$. It is easy to see that, for each sample path, a request that is lost must contribute to $\max\{X_i(K), 0\}$ for some i. Take summation and then expectation the desired result follows. Note that the amount of lost sales under the new policy is no less than that under the LDP, we have shown that $\mathbb{E}\sum_{i=1}^{I}\max\{X_i(K), 0\}$ serves an upper bound of the expected number of total lost sales under the LDP.

Recall that $\Phi(\mathbf{X}(K)) = \sum_{i=1}^{I} e^{X_i(K)/C}$. For any given constant $C > 1$, after applying Jensen's Inequality twice,

$$
e^{\frac{1}{I}\sum_{i=1}^{I}\mathbb{E}\max\{X_i(k)/C, 0\}} \le \frac{1}{I}\sum_{i=1}^{I} e^{\mathbb{E}\max\{X_i(k)/C, 0\}} \le \frac{1}{I}\sum_{i=1}^{I}\mathbb{E}e^{\max\{X_i(k)/C, 0\}}
$$

Now take logarithm on both sides of the above inequality,

$$
\sum_{i=1}^{I}\mathbb{E}\max\{X_i(K), 0\} \le IC\ln\left(\frac{1}{I}\mathbb{E}\left[\Phi(\mathbf{X}(K))\right] + 1\right).
$$

□

Proof of Lemma 3. By (2), we can write $\mathbf{X}(k+1) = \mathbf{X}(k) + \boldsymbol{\Delta}(k)$, where

$$
\Delta_i(k) = -c_i + \begin{cases} 1, \text{ if } i = i^*(j), \\ 0, \text{ otherwise,} \end{cases} \tag{EC.9}
$$

and $|\Delta_i(k)| \le 1$. Applying the inequality $e^x \le 1 + x + x^2$ ($|x| \le 1$) at $x = \Delta_i(k)/C$ ($|x| \le 1$ since $|\Delta_i(k)| \le 1$ and $C > 1$), we have

$$
e^{X_i(k+1)/C} \le e^{X_i(k)/C} + \frac{\Delta_i(k)}{C} e^{X_i(k)/C} + \frac{|\Delta_i(k)|^2}{C^2} e^{X_i(k)/C}
$$

and

$$
\begin{aligned}
&\mathbb{E}\big[\Phi\big(\mathbf{X}(k+1)\big) \mid \mathbf{X}(k)\big] - \Phi\big(\mathbf{X}(k)\big) \\
&\leq \frac{1}{C}\mathbb{E}\left[\sum_{i=1}^{I} \Delta_i(k) e^{X_i(k)/C} \middle| \mathbf{X}(k)\right] + \frac{1}{C^2}\mathbb{E}\left[\sum_{i=1}^{I} |\Delta_i(k)| e^{X_i(k)/C} \middle| \mathbf{X}(k)\right] \\
&\leq \frac{1}{C}\left(-\sum_{i=1}^{I} c_i e^{X_i(k)/C} + \sum_{j=1}^{J} p_j e^{X_{i^*(j)}(k)/C}\right) + \frac{1}{C^2}\left(\sum_{i=1}^{I} c_i e^{X_i(k)/C} + \sum_{j=1}^{J} p_j e^{X_{i^*(j)}(k)/C})\right) \\
&= \left(-\frac{1}{C^2} - \frac{1}{C}\right)\left(\sum_{i=1}^{I} c_i e^{X_i(k)/C} - \sum_{j=1}^{J} p_j e^{X_{i^*(j)}(k)/C}\right) + \frac{2}{C^2}\sum_{i=1}^{I} c_i e^{X_i(k)/C}.
\end{aligned}
$$

The last inequality follows because $|\Delta_i(k)| = c_i$ for $i \neq i^*(j)$ and $|\Delta_i(k)| \leq 1 + c_i$ for $i = i^*(j)$, and each request is of type j with probability p_j for $j \in \mathcal{J}$. □

Proof of Lemma 4. Consider the allocation rule that allocates p_j evenly to all resources in $\mathcal{I}(j)$. For any $\mathcal{J}' \subsetneq \mathcal{J}$, the resources in $\mathcal{I}(\mathcal{J}')$ can fulfill not only the request types in $\mathcal{J}'$ but also at least one request type in $\mathcal{J} \setminus \mathcal{J}'$ due to the connectivity of the network. Thus, resources in $\mathcal{I}(\mathcal{J}')$ are allocated $\sum_{j \in \mathcal{J}'} p_j$ plus $\frac{p_{\hat{j}}}{|\mathcal{I}(\hat{j})|}$ for each request type $\hat{j} \in \mathcal{J} \setminus \mathcal{J}'$ that can be fulfilled by a resource in $\mathcal{I}(\mathcal{J}')$. Hence, $\sum_{i \in \mathcal{I}(\mathcal{J}')} c_i - \sum_{j \in \mathcal{J}'} p_j \geq \frac{p_{\hat{j}}}{|\mathcal{I}(\hat{j})|} \geq \min_{j \in \mathcal{J}} \frac{p_j}{|\mathcal{I}(j)|}$ and $\eta^* \geq \min_{j \in \mathcal{J}} \frac{p_j}{|\mathcal{I}(j)|}$. □

Proof of Lemma EC.1. Since $c_i \geq \frac{\theta}{I}$ for all $i \in \mathcal{I}, \eta \geq \theta$, and $X_i(k)$ decreases in i,

$$
\begin{aligned}
&\sum_{i=1}^{\ell-1} c_i e^{X_i(k)/C} + c^{res} e^{X_\ell(k)/C} \\
\geq &\frac{\theta}{I}\sum_{i=1}^{\ell-1} e^{X_i(k)/C} + \left[\sum_{i=1}^{\ell-1}\left(c_i - \frac{\theta}{I}\right) + c^{res}\right] e^{X_\ell(k)/C} \\
= &\frac{\theta}{I}\sum_{i=1}^{\ell-1} e^{X_i(k)/C} + \left[\eta - \frac{\theta(\ell-1)}{I}\right] e^{X_\ell(k)/C} \\
\geq &\frac{\theta}{I}\sum_{i=1}^{\ell-1} e^{X_i(k)/C} + \frac{\theta(I-\ell+1)}{I} e^{X_\ell(k)/C} + (\eta-\theta) e^{X_{\min}(k)/C} \\
\geq &\frac{\theta}{I}\sum_{i=1}^{I} e^{X_i(k)/C} + (\eta-\theta) e^{X_{\min}(k)/C}.
\end{aligned}
$$

Hence by (EC.4),

$$
\Gamma = \Gamma_{(\mathcal{S},\mathcal{T})} - \eta^{\mathcal{T}} e^{X_{\min}(k)/C} \geq \frac{\theta}{I}\sum_{i=1}^{I} e^{X_i(k)/C} - \theta e^{X_{\min}(k)/C} \geq \frac{\theta}{I}\sum_{i=1}^{I} e^{X_i(k)/C} - \theta,
$$

where the last inequality follows because $X_{\min}(k) \le 0$. Similarly,

$$\begin{aligned}\sum_{i=1}^{\ell-1} c_i e^{X_i(k)/C} + c^{res} e^{X_\ell(k)/C} &\ge \sum_{i=1}^{\ell-1} c_i \eta e^{X_i(k)/C} + \left[\sum_{i=1}^{\ell-1}(c_i - c_i\eta) + c^{res}\right] e^{X_\ell(k)/C} \\ &= \eta \sum_{i=1}^{\ell-1} c_i e^{X_i(k)/C} + \eta \sum_{i=\ell}^{I} c_i e^{X_\ell(k)/C} \\ &\ge \eta \sum_{i=1}^{I} c_i e^{X_i(k)/C},\end{aligned}$$

and

$$\Gamma \ge \eta \sum_{i=1}^{I} c_i e^{X_i(k)/C} - \eta e^{X_{\min}(k)/C} \ge \eta \sum_{i=1}^{I} c_i e^{X_i(k)/C} - \eta.$$

□

Proof of Lemma EC.2. We can write $\mathbf{X}(k+1) = \mathbf{X}(k) + \ell(k)\boldsymbol{\Delta}(k)$, where

$$\Delta_i(k) = -c_i + \begin{cases} 1, \text{ if } i = i^*(j), \\ 0, \text{ otherwise,} \end{cases} \tag{EC.10}$$

and $|\Delta_i(k)| \le 1$. Applying the inequality $e^x \le 1 + x + x^2 (|x| \le 1)$ at $x = \ell(k)\Delta_i(k)/C$ ($|x| \le 1$ as $C \ge \bar{\ell}$), we have

$$e^{X_i(k+1)/C} \le e^{X_i(k)/C} + \frac{\ell(k)\Delta_i(k)}{C} e^{X_i(k)/C} + \frac{\ell^2(k)|\Delta_i(k)|^2}{C^2} e^{X_i(k)/C}$$

and

$$\begin{aligned}&\mathbb{E}\big[\Phi\big(\mathbf{X}(k+1)\big) \mid \mathbf{X}(k)\big] - \Phi\big(\mathbf{X}(k)\big) \\ &\le \frac{1}{C}\mathbb{E}\left[\sum_{i=1}^{I} \ell(k)\Delta_i(k) e^{X_i(k)/C} \middle| \mathbf{X}(k)\right] + \frac{1}{C^2}\mathbb{E}\left[\sum_{i=1}^{I} \ell^2(k)|\Delta_i(k)| e^{X_i(k)/C} \middle| \mathbf{X}(k)\right] \\ &\le \frac{1}{C}\left(-\sum_{i=1}^{I} D c_i e^{X_i(k)/C} + \sum_{j=1}^{J} p_j \mathbb{E}(\ell_j) e^{X_{i^*(j)}(k)/C}\right) \\ &\quad + \frac{1}{C^2}\left(\sum_{i=1}^{I}(\sum_{j=1}^{J} p_j \mathbb{E}(\ell_j^2)) c_i e^{X_i(k)/C} + \sum_{j=1}^{J} p_j \mathbb{E}(\ell_j^2) e^{X_{i^*(j)}(k)/C})\right) \\ &\le D\left(-\frac{A}{C^2} - \frac{1}{C}\right)\left(\sum_{i=1}^{I} c_i e^{X_i(k)/C} - \sum_{j=1}^{J} p'_j e^{X_{i^*(j)}(k)/C}\right) + \frac{2AD}{C^2}\sum_{i=1}^{I} c_i e^{X_i(k)/C}.\end{aligned}$$

The second inequality follows because $|\Delta_i(k)| = c_i$ for $i \ne i^*(j)$ and $|\Delta_i(k)| \le 1 + c_i$ for $i = i^*(j)$, and each request is of type j with probability p_j for $j \in \mathcal{J}$. The last inequality follows because of the fact $A = \max_j \frac{\mathbb{E}(\ell_j^2)}{\mathbb{E}(\ell_j)}$.

□

Proof of Lemma EC.3. Let $\bar{a}_k = \frac{\sum_{r=K-k}^{K-1} a_r}{k}$ and $\bar{a}_{\min} = \min_k \{\bar{a}_k\}$. We first use induction to show that

$$e_k \le b \sum_{i=1}^{k-1} \prod_{j=k-i}^{k-1} (1-a_j) + b \tag{EC.11}$$

for all $k \ge 1$. Since $e_1 \le b$, the conclusion holds for $k=1$. Suppose that the conclusion holds for k. For the case of $k+1$,

$$\begin{aligned} e_{k+1} &\le (1-a_k)e_k + b \le (1-a_k)\left(b\sum_{i=1}^{k-1}\prod_{j=k-i}^{k-1}(1-a_j)+b\right)+b \\ &= b\sum_{i=1}^{k-1}\prod_{j=k-i}^{k}(1-a_j)+b(1-a_k)+b = b\sum_{i=1}^{k}\prod_{j=k+1-i}^{k}(1-a_j)+b. \end{aligned}$$

Having established (EC.11), we can reach the desired result by the arithmetic-mean and geometric-mean inequality:

$$e_K \le b\sum_{i=1}^{K-1}\prod_{j=K-i}^{K-1}(1-a_j)+b \le b\sum_{i=1}^{K-1}(1-\bar{a}_i)^i + b \le b\sum_{i=1}^{K-1}(1-\bar{a}_{\min})^i + b \le b/\bar{a}_{\min}.$$

□