




Instantaneous Control of Brownian Motion with a Positive Lead Time

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Abstract. Consider a storage system where the content is driven by a Brownian motion in the absence of control. At any time, one may increase or decrease the content at a cost proportional to the amount of adjustment. A decrease of the content takes effect immediately, while an increase is realized after a fixed lead time ℓ . Holding costs are incurred continuously over time and are a convex function of the content. The objective is to find a control policy that minimizes the expected present value of the total costs. Because of the positive lead time for upward adjustments, one needs to keep track of all of the outstanding upward adjustments as well as the actual content at time t as there may also be downward adjustments during $[t, t + \ell]$ —that is, the state of the system is a function on $[0, \ell]$. We first extend the concept of L^h -convexity to function spaces and establish the L^h -convexity of the optimal cost function. We then derive various properties of the cost function and identify the structure of the optimal policy as a state-dependent two-sided reflection mapping making the minimum amount of adjustment necessary to keep the system states within a certain region.

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1. Introduction

Consider a storage system, such as an inventory or cash fund, whose content fluctuates according to a Brownian motion in the absence of control. A convex holding cost of the content is incurred continuously. At any time, a controller may initiate an upward adjustment to increase the content, which is realized after a lead time, and/or a downward adjustment to decrease the content, which takes effect immediately. Both upward and downward adjustments incur a variable cost. The objective is to find a control policy that minimizes the expected discounted cost over an infinite planning horizon.

In the absence of the lead time, the state of the problem is one dimensional, and Harrison and Taksar [10, 11] show that an optimal control policy can be characterized by two closed-form control limits. The method used to analyze the problem is referred to as a lower-bound approach by Dai and Yao [5], and involves three steps. (1) Based on the optimality equations, heuristically derive some differential inequalities of the optimal cost function, with at least one equation being tight. This is known as the Hamilton–Jacobi–Bellman (HJB) equation. (2) For a control limit policy, first obtain a set of ordinary differential equations (ODEs) of the cost function and then solve those equations. (3) Find the control limits under which the cost function is continuously differentiable and hence optimal.

The problem becomes much more complicated, however, when there is a positive lead time ℓ for upward adjustments. This is because the on-hand inventory at $t + \ell$ cannot be predicted solely from the inventory position at any time t as there may be downward adjustments in $[t, t + \ell]$. One needs to keep track of the amount and timing of each outstanding upward adjustment as well as the content on hand at any time—that is, the state of the system is a function on $[0, \ell]$. Thus, step (2) of the lower bound approach will only result in an uncountable number of partial differential equations (PDEs) with unknown boundary conditions, which are almost impossible to solve.

To derive and prove the structure of the optimal control policy in the presence of a positive lead time, we follow step (1) to heuristically derive an HJB equation based on two optimality conditions, optimizing the timing and amounts of adjustments, respectively. The similarity between our analysis and the lower bound approach in Harrison and Taksar [11] stops here, and we proceed with the following steps, each of which

involves challenging and deep mathematical analysis. (2) Extend the concept of L^h -convexity defined on finite dimensional spaces and introduced by Murota [13] to a function space, and show that the optimal cost function is the limit of the costs of a series of periodic review systems and hence is L^h -convex in our state space. This is one of the key steps in our analysis and a fundamental building block. (3) Derive some properties of the optimal cost function using the L^h -convexity of the cost function, and identify two sets of states in which an upward and a downward adjustment are needed, respectively. These two sets also reveal the boundaries of the PDEs for the HJB equation. (4) Construct a state-dependent two-sided reflection policy making the minimum amount of upward or downward adjustment necessary to prevent the state from entering into the two sets and prove it is optimal. Such a policy is much more complicated than that in Harrison and Taksar [11], and the proof of its optimality requires the establishment of properties such as the monotonicity, Lipschitz continuity, and complementarity of the policy.

Existing methods can only deal with systems with single dimensional states—for example, zero lead time for both upward and downward adjustments in our problem. For periodic control problems, except for those with states of one or two dimensions, the common approach is to establish the L^h -convexity of the optimal cost function, with which a threshold policy can be easily shown to be optimal. Such an approach cannot be applied directly to problems with instantaneous control as L^h -convexity is only defined on finite dimensional spaces. As one can see, identifying the optimal policy is nontrivial even after extending and applying the concept of L^h -convexity to a function space (i.e., step (2)), and requires additional challenging steps—that is, steps (3) and (4) mentioned above.

The remainder of this paper is organized as follows. In Section 2, we provide a brief summary of relevant literature. In Section 3, we present a precise mathematical formulation of the Brownian control problem. We then derive two optimality conditions and provide a heuristic derivation of the HJB equations. In Section 4, we extend the concept of L^h -convexity to a function space, and show that the optimal cost function is the limit of the costs of a series of periodic review systems and hence is L^h -convex. In Section 5, we provide various properties of the optimal cost function, which lead to the optimal control being a state-dependent two-sided reflection policy in Section 6. The conclusion and discussion are included in Section 7.

2. Literature Review

Research on the stochastic control of Brownian motion dates back to Bather [1] and the early work was aimed at minimizing the total expected discounted costs. Constantinides and Richard [4] show that a control band policy is optimal when there is a fixed cost for upward and downward adjustments, and Harrison et al. [12] develop a method to find the optimal bands. Davis [7] and Øksendal and Sulem [14] show the equivalence of this control problem to a sequence of optimal stopping problems. All of these papers assume that the holding cost is linear. Dai and Yao [6] extend this work to a general convex holding cost function. Harrison and Taksar [10, 11] prove that a control limit policy is optimal absent fixed costs under linear and convex holding costs, respectively, and the latter also provides a procedure for computing the optimal limits. The methodology used in these papers is the three-step approach described in the introduction. Later, these policies are shown to be optimal also under the average cost criteria by Ormeci et al. [15] and Dai and Yao [5] with fixed costs when the holding cost is linear and convex, respectively, and by Taksar [22] without a fixed cost.

Note that all of the abovementioned work assumes away a positive lead time for upward or downward adjustments, except that of Øksendal and Sulem [14], which shows that, with some additional assumptions that will be discussed in Section 7, the problem where the lead times for upward and downward adjustments are the same can be reduced to one with zero lead times.

Since the state in our problem is on a function space, the literature on L^h -convexity that extends convexity to multiple dimensions is also relevant. We refer to Zipkin [24] for an excellent summary of the development of the concept and its application in inventory management. By establishing the L^h -convexity of the optimal cost function, Zipkin [24] develops a new approach to the structural analysis of the standard, single-item, lost-sales inventory system with a linear ordering cost and a positive replenishment lead time. This concept is also used in the structural analysis of problems where the state is of a finite dimension—for example, inventory-pricing control with lead times (Pang et al. [16]) and perishable inventory systems (Chen et al. [3]). In our paper, we will extend L^h -convexity to a function space.

The two-sided reflection policy shown to be optimal for our problem is inspired by the work of Skorokhod [20, 21], which solves the stochastic differential equation for a reflecting Brownian motion. The idea of the reflection mapping is widely used in the study of queueing systems. For example, Harrison and Reiman [9] and Reiman [19] obtain the heavy-traffic limits for some open queueing network using multidimensional reflection mappings. We refer to Chen and Yao [2] and Whitt [23] for more in-depth knowledge about reflection mappings.

3. Model Description

In this section, we formulate the problem mathematically and heuristically derive the Hamilton–Jacobi–Bellman (HJB) equation.

3.1. Problem Formulation

3.1.1. Modeling Details. Let Ω be the set of all continuous functions $\omega: [0, \infty) \rightarrow \mathbb{R}$, and $W_t: \Omega \rightarrow \mathbb{R}$ be the coordinate projection map $W_t(\omega) = \omega(t)$ for $t \geq 0$. Denote by $\mathcal{F} = \sigma(W_t, t \geq 0)$ the smallest σ -field such that W_t is \mathcal{F} -measurable and $\mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t)$ for each $t \geq 0$. Also let \mathbb{P} be the unique probability measure on (Ω, \mathcal{F}) such that W_t is a Brownian motion with drift μ and variance σ^2 , and \mathbb{E} be the associated expectation operator.

Now consider a storage system, such as an inventory or bank account, whose content H_t , $t \geq 0$, fluctuates according to a Brownian motion W_t with drift μ and variance σ^2 , without any control. Holding costs are incurred continuously at the rate $h(H_t)$. At any time, we may take an action to cause the storage level to jump by a positive amount after a fixed lead time ℓ or by a negative amount that takes effect immediately. An upward adjustment incurs a variable cost k^\uparrow , while a downward adjustment incurs a variable cost k^\downarrow . Thus, the cost for an upward ξ^\uparrow and/or downward ξ^\downarrow adjustment at any given time is given by

$$\phi(\xi^\uparrow, \xi^\downarrow) = k^\uparrow \xi^\uparrow + k^\downarrow \xi^\downarrow. \tag{1}$$

When $\ell = 0$, the problem reduces to that in Harrison and Taksar [11]. With a positive lead time for upward adjustments, the problem becomes much more complicated for the following reasons. (i) As instantaneous downward adjustments can occur at any time, by itself the *inventory position* at any time t cannot predict the content on hand and hence the expected holding cost at time $t + \ell$. One needs to keep track of the content at any time t , as well as all of the upward adjustments that will be realized in $[t, t + \ell)$, or a profile of outstanding upward adjustments. (ii) With continuous time, such a profile is a function on $[0, \ell]$. Dynamic control with infinite dimensional state variables is well known to be extremely challenging, and there has been little work in the literature. Next, we define the state and decision variables, and provide the system dynamics of the model.

1. The state variables: Let $\mathcal{X}_t(u) \in \mathbb{R}$ be the content of the system plus the total amount of outstanding upward adjustments at time t that will be realized by $t + u$. Then, $\mathcal{X}_t(0)$ is simply the content of the system at time t . Since all of the outstanding upward adjustments at time t will be realized before $t + \ell$, we have $\mathcal{X}_t(u) = \mathcal{X}_t(\ell)$ for $u > \ell$ in our state. Thus, $\mathcal{X}_t(u), u \geq 0$, is right-continuous, nondecreasing, and constant for $u \geq \ell$.

Let $\mathcal{X}_t = \{\mathcal{X}_t(u), u \geq 0\}$ be the state of the system at time t and \mathbb{D} be the set of all possible states. That is, \mathbb{D} is the set of all functions on \mathbb{R}_+ with the following properties: (1) right-continuous on $[0, \infty)$ with left limits in $(0, \infty)$, (2) nondecreasing, and (3) complete, which we establish in the appendix. For convenience, we denote $\mathcal{F} = \{\mathcal{F}(u) = 1, u \geq 0\} \in \mathbb{D}$ and $\mathcal{X} + a = \{\mathcal{X}(u) + a, u \geq 0\} \in \mathbb{D}$ for $a \in \mathbb{R}$.

2. The decision variables: Let $Y^\uparrow(t)$ and $Y^\downarrow(t)$ be stochastic processes adapted to the filtration \mathcal{F}_t for all $t \geq 0$, representing the cumulative upward and downward adjustments up to time t , respectively. Thus, Y^\uparrow and Y^\downarrow are nondecreasing functions. For convenience, let $\pi = (Y^\uparrow, Y^\downarrow) = \{(Y^\uparrow(t), Y^\downarrow(t)) : t \geq 0\}$ represent a control policy over the planning horizon such that any control at time t is based on information that has been revealed up to t .

3. The system dynamics: For any $t > 0$, given the initial state \mathcal{X}_0 and policy $\pi = (Y^\uparrow, Y^\downarrow)$, we have that

$$\mathcal{X}_t(u) = \begin{cases} \mathcal{X}_0(u+t) + Y^\uparrow(t+u-\ell) - Y^\downarrow(t) + W_t & u \leq \ell, \\ \mathcal{X}_t(\ell) & u > \ell. \end{cases} \tag{2}$$

To see the intuition, we just need to focus on u where $u \leq \ell$. Apart from W_t , $\mathcal{X}_t(u)$ includes the content at time 0 plus the upward adjustments made before time $t + u - \ell$, minus the downward adjustments made up to t . The content at time 0 plus the upward adjustments made before time $t + u - \ell$ consists of two parts: (1) $\mathcal{X}_0(u+t)$ is the content of the system at time 0 plus the upward adjustments made before time 0 that will be realized by $t + u$; (2) $Y^\uparrow(t+u-\ell)$ is cumulative upward adjustments made after time 0 that will be realized by $t + u$. Taking $u = 0$ in (2), the content on hand at t can be written as $H_t = \mathcal{X}_t(0) = \mathcal{X}_0(t) + Y^\uparrow(t-\ell) - Y^\downarrow(t) + W_t$.

3.1.2. The Cost Function. For any given policy π and initial state $\mathcal{X} = \mathcal{X}_0 \in \mathbb{D}$, the total expected cost can be written as

$$C(\mathcal{X}, \pi) = \mathbb{E} \left[\int_0^\infty e^{-\gamma t} h(\mathcal{X}_t(0)) dt + \int_0^\infty e^{-\gamma t} (k^\uparrow dY^\uparrow(t) + k^\downarrow dY^\downarrow(t)) \right], \tag{3}$$

where γ is the discount rate. We impose the following mild assumptions on the holding cost function for the rest of this paper.

Assumption 1. *The holding cost function $h: \mathbb{R} \rightarrow \mathbb{R}^+$ satisfies the following conditions: (1) $h(\cdot)$ is convex and piecewise C^2 -continuous; (2) $h(0) = 0$; and (3) there exists $M > 0$ such that $|h'(\cdot)| \leq M$.*

Parts (1) and (2) of Assumption 1 guarantee that it is never optimal to make a downward adjustment exceeding the available content at any time. Without loss of generality, we will only consider feasible policies that result in finite control costs—that is,

$$\mathbb{E} \left[\int_0^\infty e^{-\gamma t} (dY^\uparrow(t) + dY^\downarrow(t)) \right] < \infty. \tag{4}$$

Thus, under Assumption 1, a policy π is feasible if and only if $C(\mathcal{X}, \pi)$ is finite. Denote by Π the set of all such control policies and by $C^*(\mathcal{X}) = \inf_{\pi \in \Pi} \{C(\mathcal{X}, \pi)\}$ the optimal cost.

The following proposition establishes that the optimal cost $C^*(\mathcal{X})$ is Lipschitz continuous on \mathbb{D} . All of the proofs in the paper are either in the main body or can be found in the appendix. Since the states are functions, we define the distance between two states \mathcal{X} and $\mathcal{X}' \in \mathbb{D}$ as $d(\mathcal{X}, \mathcal{X}') = \int_0^\infty e^{-\gamma t} |\mathcal{X}(t) - \mathcal{X}'(t)| dt$. It is easy to see that the space \mathbb{D} is a complete metric space under the distance $d(\cdot, \cdot)$.

Proposition 1. *Under Assumption 1, $C^*(\mathcal{X})$ is Lipschitz continuous. That is, for any states \mathcal{X} and \mathcal{X}' , $|C^*(\mathcal{X}) - C^*(\mathcal{X}')| \leq Md(\mathcal{X}, \mathcal{X}')$.*

3.2. Heuristic Derivation of the Hamilton–Jacobi–Bellman (HJB) Equations

We first note that, for any given initial state $\mathcal{X} \in \mathbb{D}$, the optimal cost should satisfy the following optimality conditions:

$$C^*(\mathcal{X}) = \inf_{\xi^\uparrow \geq 0, \xi^\downarrow \geq 0} \{ \phi(\xi^\uparrow, \xi^\downarrow) + C^*(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X})) \}, \tag{5}$$

$$C^*(\mathcal{X}) = \inf_{s \geq 0} \left\{ \mathbb{E} \left[\int_0^s e^{-\gamma u} h(\mathcal{X}(u) + W_u) du + e^{-\gamma s} C^*(\sigma_s(\mathcal{X}) + W_s) \right] \right\}, \tag{6}$$

where s is a stopping time and

$$\begin{aligned} \Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}) &= \{ \mathcal{X}(u) - \xi^\downarrow + \xi^\uparrow \mathbf{1}_{\{u \geq \ell\}} : u \geq 0 \}, \\ \sigma_s(\mathcal{X}) &= \{ \mathcal{X}(s + u), u \geq 0 \} \end{aligned} \tag{7}$$

are the states after an adjustment $(\xi^\uparrow, \xi^\downarrow)$ is made and after a period of time s with no adjustment for a given initial state \mathcal{X} , respectively. Let

$$C(\mathcal{X}, \xi^\uparrow, \xi^\downarrow) = \phi(\xi^\uparrow, \xi^\downarrow) + C^*(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X})) \tag{8}$$

be the minimum cost under a given adjustment $(\xi^\uparrow, \xi^\downarrow)$. Assume for now that $\frac{\partial C(\mathcal{X}, \xi^\uparrow, \xi^\downarrow)}{\partial \xi^\uparrow}$ and $\frac{\partial C(\mathcal{X}, \xi^\uparrow, \xi^\downarrow)}{\partial \xi^\downarrow}$ exist, which we will prove later. Then, with a small amount of adjustment ϵ ,

$$C(\mathcal{X}, \epsilon, 0) = C^*(\mathcal{X}) + \frac{\partial C(\mathcal{X}, 0, 0)}{\partial \xi^\uparrow} \epsilon + o(\epsilon), \tag{9}$$

$$C(\mathcal{X}, 0, \epsilon) = C^*(\mathcal{X}) + \frac{\partial C(\mathcal{X}, 0, 0)}{\partial \xi^\downarrow} \epsilon + o(\epsilon). \tag{10}$$

If $(\xi^\uparrow, \xi^\downarrow) = (0, 0)$ (i.e., no adjustment is made at time 0), absent further adjustment, the state at time $s > 0$ becomes $\sigma_s(\mathcal{X}) + w$ for any realization of $W_s = w$. We define

$$V_{\mathcal{X}}(w, s) = C^*(\sigma_s(\mathcal{X}) + w). \tag{11}$$

If no adjustment is made for ϵ amount of time, then, by Ito’s formula, the minimum expected discounted cost becomes

$$\mathbb{E} \left[\int_0^\epsilon e^{-\gamma t} h(\mathcal{X}_t(0)) dt + e^{-\gamma \epsilon} V_{\mathcal{X}}(W_\epsilon, \epsilon) \right] = C^*(\mathcal{X}) + [\Gamma V_{\mathcal{X}}(0, 0) - \gamma V_{\mathcal{X}}(0, 0) + h(\mathcal{X}(0))] \epsilon + o(\epsilon), \tag{12}$$

where the operator $\Gamma = \frac{\partial}{\partial s} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial w^2} + \mu \frac{\partial}{\partial w}$. Thus, for any given \mathcal{X} ,

$$[\Gamma V_{\mathcal{X}}(0,0) - \gamma V_{\mathcal{X}}(0,0) + h(\mathcal{X}(0))] \wedge \frac{\partial C(\mathcal{X},0,0)}{\partial \xi^\uparrow} \wedge \frac{\partial C(\mathcal{X},0,0)}{\partial \xi^\downarrow} = 0, \tag{13}$$

which is precisely the HJB equation associated with the state \mathcal{X} . Since there is an uncountable number of \mathcal{X} in the state space, the HJB equations involve an uncountable number of PDEs with unknown boundary conditions, and no known method is available to solve it directly. Instead, we will solve the problem by first establishing the L^\natural -convexity of the optimal cost function on function spaces. A solution to the HJB equations will be given in Theorem 4.

4. The L^\natural -Convexity of the Optimal Cost Function

Since the concept of L^\natural -convexity is defined on \mathbb{R}^n , we first study a periodic version of the problem. We then extend the concept of L^\natural -convexity from \mathbb{R}^n to \mathbb{D} by linking the problem to the limit of a series of periodic problems.

4.1. A Periodic Review System

Consider a periodic review of the system with period length $\frac{\ell}{n}$ —that is, an upward adjustment takes n periods. In such a system, the state in any period is an n -dimensional vector denoted by $\mathbf{x}_t = (x_{t,0}, x_{t,1}, \dots, x_{t,n-1})$ where $x_{t,0}$ is the current content of the system and $x_{t,i}$, $1 \leq i \leq n-1$, is the content of the system plus the total outstanding upward movement that will be realized from period $t+1$ to $t+i$. Letting y_t^\uparrow and y_t^\downarrow be the upward and downward adjustments in period t , we obtain the following dynamics:

$$\mathbf{x}_{t+1} = (x_{t,1}, x_{t,2}, \dots, x_{t,n-1}, x_{t,n-1} + y_t^\uparrow) - y_t^\downarrow \mathbf{e} + w_t \mathbf{e}, \tag{14}$$

where \mathbf{e} is a vector of all 1's whose dimension will be clear from the context, and $w_t = W_{\frac{(t+1)\ell}{n}} - W_{\frac{t\ell}{n}}$ is the random change caused by the Brownian motion. Let \mathcal{N}_a represent a normally distributed random variable with mean $a\mu$ and variance $a\sigma^2$ for any $a > 0$. Then, the discount rate becomes $\alpha = e^{-\gamma \frac{\ell}{n}}$ and holding cost is given by

$$h^n(x) = \mathbb{E} \left[\int_0^{\frac{\ell}{n}} e^{-\gamma s} h \left(x + \mathcal{N}_{\frac{\ell}{n}} \right) ds \right]$$

in the periodic system.

Next, we present some definitions where the concept of L^\natural -convexity can be found in Zipkin [24], and show that the optimal cost function for the periodic system is L^\natural -convex.

Definition 1. Let f be a function on \mathbb{R}^n .

1. f is submodular if for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$, $f(\mathbf{x}_1) + f(\mathbf{x}_2) \geq f(\mathbf{x}_1 \vee \mathbf{x}_2) + f(\mathbf{x}_1 \wedge \mathbf{x}_2)$.
2. f is L^\natural -convex if the function $g(\mathbf{x}, \xi) = f(\mathbf{x} - \xi \mathbf{e})$ is submodular in \mathbb{R}^{n+1} .

Thus, a function f is L^\natural -convex if and only if, for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and $\xi_1, \xi_2 \in \mathbb{R}$,

$$f(\mathbf{x}_1 - \xi_1 \mathbf{e}) + f(\mathbf{x}_2 - \xi_2 \mathbf{e}) \geq f(\mathbf{x}_1 \vee \mathbf{x}_2 - (\xi_1 \vee \xi_2) \mathbf{e}) + f(\mathbf{x}_1 \wedge \mathbf{x}_2 - (\xi_1 \wedge \xi_2) \mathbf{e}).$$

To show the L^\natural -convexity of the optimal cost function for the periodic system, we define $C_t^{T,n}(\mathbf{x}_t)$ as the optimal cost function from period t to T for a given (T, n) and state \mathbf{x}_t . Then,

$$C_t^{T,n}(\mathbf{x}_t) = \min_{y_t^\uparrow, y_t^\downarrow \geq 0} \{c_t^{T,n}(\mathbf{x}_t, y_t^\uparrow, y_t^\downarrow)\},$$

where

$$c_t^{T,n}(\mathbf{x}_t, y_t^\uparrow, y_t^\downarrow) = k^\uparrow y_t^\uparrow + k^\downarrow y_t^\downarrow + \alpha \mathbb{E} [C_{t+1}^{T,n}(\mathbf{x}_{t+1}) + h^n(x_{t,0} - y_t^\downarrow)]$$

for $0 \leq t \leq T-1$ and $C_T^{T,n}(\mathbf{x}_T) = 0$.

Proposition 2. $c_t^{T,n}(\mathbf{x}, y^\uparrow, y^\downarrow)$ is L^\natural -convex in $(\mathbf{x}, x_n, y^\downarrow)$, and $C_t^{T,n}(\mathbf{x})$ is L^\natural -convex in \mathbf{x} .

By theorem 6.2.3 of Puterman [18], $C^\infty(\mathbf{x}) := \lim_{T \rightarrow \infty} \{C_0^{T,n}(\mathbf{x})\} < \infty$ is the unique solution to the optimality equation $C^\infty(\mathbf{x}) = \min_{y^\uparrow, y^\downarrow \geq 0} \{c^\infty(\mathbf{x}, y^\uparrow, y^\downarrow)\}$ where

$$c^\infty(\mathbf{x}, y^\uparrow, y^\downarrow) = k^\uparrow y^\uparrow + k^\downarrow y^\downarrow + \alpha \mathbb{E} [C^\infty((x_1, x_2, \dots, x_{n-1}, x_{n-1} + y^\uparrow) - y^\downarrow \mathbf{e} + w_t \mathbf{e}) + h^n(x_0 - y^\downarrow)]$$

and hence we have the following theorem.

Theorem 1. $C^{\infty,n}(\mathbf{x})$ is L^{\natural} -convex and hence the optimal cost for the infinite horizon periodic review system for any given n .

Thus, there exists a unique optimal adjustment $(y^{\uparrow}, y^{\downarrow})$ for any given \mathbf{x} and the optimal y^{\uparrow} (y^{\downarrow}) is increasing (decreasing) in \mathbf{x} , where the order of \mathbf{x} in \mathbb{R}^n is defined in the usual way of component-wise comparison.

4.2. The Continuous Review System

Since the state \mathcal{X} is defined on \mathbb{D} rather than \mathbb{R}^n , we need to extend the concept of L^{\natural} -convexity to \mathbb{D} . The L^{\natural} -convexity of $C^*(\mathcal{X})$ will enable us to construct an optimal policy in Section 6.

Definition 2. Suppose that $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{D}$.

- Order: $\mathcal{X}_1 \geq \mathcal{X}_2$ if $\mathcal{X}_1(u) \geq \mathcal{X}_2(u)$ for any $u \geq 0$, and $\mathcal{X}_1 \leq \mathcal{X}_2$ if $\mathcal{X}_1(u) \leq \mathcal{X}_2(u)$ for any $u \geq 0$.
- Max and Min Operations: $\mathcal{X}_1 \vee \mathcal{X}_2 = \{\mathcal{X}_1(u) \vee \mathcal{X}_2(u), u \geq 0\}$ and $\mathcal{X}_1 \wedge \mathcal{X}_2 = \{\mathcal{X}_1(u) \wedge \mathcal{X}_2(u), u \geq 0\}$.

Definition 3. A function F on \mathbb{D} is L^{\natural} -convex if, for any $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{D}$ and $\xi_1, \xi_2 \in \mathbb{R}$,

$$F(\mathcal{X}_1 - \xi_1) + F(\mathcal{X}_2 - \xi_2) \geq F(\mathcal{X}_1 \vee \mathcal{X}_2 - (\xi_1 \vee \xi_2)) + F(\mathcal{X}_1 \wedge \mathcal{X}_2 - (\xi_1 \wedge \xi_2)).$$

To connect the periodic review systems with our original one, for any given state \mathcal{X} and policy π , consider the following discretized state \mathcal{X}^n and policy π^n that makes adjustments only at multiples of $\frac{\ell}{n}$. It is easy to see that \mathcal{X}^n and π^n approach \mathcal{X} pointwise and π , respectively, as $n \rightarrow \infty$.

1. The state \mathcal{X}^n is such that

$$\mathcal{X}^n(u) = \begin{cases} \mathcal{X}(\frac{\ell}{n}), & \text{if } 0 \leq u \leq \frac{\ell}{n}, \\ \mathcal{X}(\frac{i\ell}{n}), & \text{if } \frac{(i-1)\ell}{n} < u \leq \frac{i\ell}{n}, i = 2, 3, \dots, n, \\ \mathcal{X}(\ell), & \text{if } u > \ell. \end{cases} \tag{15}$$

Let

$$\mathbf{x}^n = \left(\mathcal{X}\left(\frac{\ell}{n}\right), \mathcal{X}\left(\frac{2\ell}{n}\right), \dots, \mathcal{X}\left(\frac{(n-1)\ell}{n}\right), \mathcal{X}(\ell) \right).$$

2. The policy $\pi^n = (Y^{n\uparrow}, Y^{n\downarrow})$ is such that

$$Y^{n\uparrow}(t) = \sum_{i=0}^{\lfloor \frac{nt}{\ell} \rfloor} \xi_i^{n\uparrow} \text{ and } Y^{n\downarrow}(t) = \sum_{i=0}^{\lfloor \frac{nt}{\ell} \rfloor} \xi_i^{n\downarrow}, \tag{16}$$

where $(\xi_0^{n\uparrow}, \xi_0^{n\downarrow}) = (Y^{\uparrow}(0), Y^{\downarrow}(0))$ and $(\xi_i^{n\uparrow}, \xi_i^{n\downarrow}) = \left(Y^{\uparrow}(\frac{i\ell}{n}) - Y^{\uparrow}(\frac{(i-1)\ell}{n}), Y^{\downarrow}(\frac{i\ell}{n}) - Y^{\downarrow}(\frac{(i-1)\ell}{n}) \right)$ for $i = 1, 2, \dots$.

Then, the cost of the system for a given (\mathcal{X}^n, π^n) is given by

$$C(\mathcal{X}^n, \pi^n) = \mathbb{E} \left[\int_0^{\infty} e^{-\gamma t} h(\mathcal{X}_t^n(0)) dt + \int_0^{\infty} e^{-\gamma t} (k^{\uparrow} dY^{n\uparrow}(t) + k^{\downarrow} dY^{n\downarrow}(t)) \right], \tag{17}$$

where $\mathcal{X}_t^n(\cdot)$ is the corresponding state at time t under π^n with the initial state \mathcal{X}^n . By (2), we also have $\mathcal{X}_t^n(0) \rightarrow \mathcal{X}_t(0)$ as $n \rightarrow \infty$ for any $t \geq 0$. It then follows, by (3), (17) and the Lebesgue’s dominated convergence theorem, that

$$\lim_{n \rightarrow \infty} C(\mathcal{X}^n, \pi^n) = C(\mathcal{X}, \pi). \tag{18}$$

It remains to be shown that the optimal cost of the original problem is the limit of the costs of periodic review systems and hence is L^{\natural} -convexity by Theorem 1.

Proposition 3. $C^*(\mathcal{X}) = \lim_{n \rightarrow +\infty} C^{\infty,n}(\mathbf{x}^n)$.

Proof of Proposition 3. Since $C^*(\mathcal{X})$ is the optimal cost, for any $\epsilon > 0$, we can find a policy π such that $C(\mathcal{X}, \pi) < C^*(\mathcal{X}) + \epsilon$. On the other hand, as $C^{\infty, n}(\cdot)$ is the optimal cost of the periodic review system, $C^{\infty, n}(\mathbf{x}^n) \leq C(\mathcal{X}^n, \pi^n)$. Then, we have

$$\limsup_{n \rightarrow +\infty} C^{\infty, n}(\mathbf{x}^n) \leq \lim_{n \rightarrow +\infty} C(\mathcal{X}^n, \pi^n) = C(\mathcal{X}, \pi) < C^*(\mathcal{X}) + \epsilon.$$

As $\epsilon > 0$ is arbitrary, we have $\limsup_{n \rightarrow +\infty} C^{\infty, n}(\mathbf{x}^n) \leq C^*(\mathcal{X})$. Combined with the fact that $\liminf_{n \rightarrow +\infty} C^{\infty, n}(\mathbf{x}^n) \geq \lim_{n \rightarrow +\infty} C(\mathcal{X}^n) \geq C^*(\mathcal{X})$, we have the result. \square

Theorem 2. *The optimal cost $C^*(\mathcal{X})$ is L^{\natural} -convex in \mathbb{D} .*

Proof of Theorem 2. For any $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{D}$ and their respective \mathbf{x}_1^n and \mathbf{x}_2^n , it is clear that $\mathbf{x}_1^n \vee \mathbf{x}_2^n$ is the vector form of $(\mathcal{X}_1 \vee \mathcal{X}_2)^n = (\mathcal{X}_1)^n \vee (\mathcal{X}_2)^n$ and $\mathbf{x}_1^n \wedge \mathbf{x}_2^n$ is the vector form of $(\mathcal{X}_1 \wedge \mathcal{X}_2)^n = (\mathcal{X}_1)^n \wedge (\mathcal{X}_2)^n$. For any $\xi_1, \xi_2 \in \mathbb{R}$, by the L^{\natural} -convexity of $C^{\infty, n}(\mathbf{x})$ in Theorem 1,

$$C^{\infty, n}(\mathbf{x}_1^n - \xi_1 \mathbf{e}) + C^{\infty, n}(\mathbf{x}_2^n - \xi_2 \mathbf{e}) \geq C^{\infty, n}(\mathbf{x}_1^n \vee \mathbf{x}_2^n - (\xi_1 \vee \xi_2) \mathbf{e}) + C^{\infty, n}(\mathbf{x}_1^n \wedge \mathbf{x}_2^n - (\xi_1 \wedge \xi_2) \mathbf{e}).$$

Letting $n \rightarrow \infty$, we see that $C^*(\mathcal{X})$ satisfies Definition 3. \square

5. Properties of the Optimal Cost Function $C^*(\mathcal{X})$

5.1. Impact of Adjustments on the Cost Function

Recall the function $C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})$ and their partial derivatives $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}}$ and $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\downarrow}}$ introduced in Section 3.2. A quick fact is that the L^{\natural} -convexity of $C^*(\mathcal{X})$ immediately implies that the cost function $C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})$ is convex and differentiable in ξ^{\uparrow} and ξ^{\downarrow} . The following properties of the partial derivatives will help identify the control regions and consequently construct the optimal policy in Section 6.

Lemma 1. *Monotonicity of the derivatives:*

1. If $\mathcal{X}_1 \leq \mathcal{X}_2$ and $\mathcal{X}_1(\ell) = \mathcal{X}_2(\ell)$, $\frac{\partial C(\mathcal{X}_1, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}} \geq \frac{\partial C(\mathcal{X}_2, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}}$.
2. For $a > 0$, $\frac{\partial C(\mathcal{X}+a, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}} \geq \frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}}$.
3. $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}}$ is decreasing in \mathcal{X} .

Lemma 2. *Continuity of the derivatives: $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}}$ and $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\downarrow}}$ are continuous in \mathcal{X} .*

Proof of Lemma 2. Since the proofs are similar, we only prove the continuity for $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}}$. Suppose it is not continuous and there exists $a_0 > 0$ and a sequence $\{\mathcal{X}_n, n = 1, 2, \dots\}$ in \mathbb{D} such that, as $n \rightarrow \infty$, $d(\mathcal{X}, \mathcal{X}_n) \rightarrow 0$ but $\frac{\partial C(\mathcal{X}_n, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}} - \frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}} > 2a_0$ or $< -2a_0$ for all n . Since $C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})$ is convex in ξ^{\uparrow} , there exists $b_0 > 0$ such that $\frac{\partial C(\mathcal{X}, \xi^{\uparrow} + b_0, \xi^{\downarrow})}{\partial \xi^{\uparrow}} < \frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}} + a_0$. Thus, $\frac{\partial C(\mathcal{X}_n, \xi^{\uparrow} + b_0, \xi^{\downarrow})}{\partial \xi^{\uparrow}} < \frac{\partial C(\mathcal{X}_n, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}} - a_0$. Since $C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})$ is convex in ξ^{\uparrow} , the partial derivative $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}}$ is increasing in ξ^{\uparrow} . We have

$$\begin{aligned} C(\mathcal{X}, \xi^{\uparrow} + b_0, \xi^{\downarrow}) - C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow}) &= \int_0^{b_0} \frac{\partial C(\mathcal{X}, \xi^{\uparrow} + s, \xi^{\downarrow})}{\partial \xi^{\uparrow}} ds \leq \int_0^{b_0} \frac{\partial C(\mathcal{X}, \xi^{\uparrow} + b_0, \xi^{\downarrow})}{\partial \xi^{\uparrow}} ds \\ &\leq \int_0^{b_0} \left(\frac{\partial C(\mathcal{X}_n, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}} - a_0 \right) ds \leq \int_0^{b_0} \frac{\partial C(\mathcal{X}_n, \xi^{\uparrow} + s, \xi^{\downarrow})}{\partial \xi^{\uparrow}} ds - \int_0^{b_0} a_0 ds \\ &= C(\mathcal{X}_n, \xi^{\uparrow} + b_0, \xi^{\downarrow}) - C(\mathcal{X}_n, \xi^{\uparrow}, \xi^{\downarrow}) - a_0 b_0. \end{aligned}$$

On the other hand, because $d(\mathcal{X}, \mathcal{X}_n) \rightarrow 0$ as $n \rightarrow \infty$ and $C^*(\mathcal{X})$ is continuous in \mathbb{D} , $C(\mathcal{X}_n, \xi^{\uparrow} + b_0, \xi^{\downarrow}) - C(\mathcal{X}_n, \xi^{\uparrow}, \xi^{\downarrow})$ converges to $C(\mathcal{X}, \xi^{\uparrow} + b_0, \xi^{\downarrow}) - C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})$ as $n \rightarrow \infty$. This is a contradiction, and $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}}$ is continuous in \mathcal{X} . \square

We also note that $C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow}) = C^*(\Phi_{\xi^{\uparrow}, \xi^{\downarrow}}(\mathcal{X})) \leq C^*(\Phi_{\xi^{\uparrow} + \epsilon^{\uparrow}, \xi^{\downarrow} + \epsilon^{\downarrow}}(\mathcal{X})) + \phi(\xi^{\uparrow} + \epsilon^{\uparrow}, \xi^{\downarrow} + \epsilon^{\downarrow}) = C^*(\mathcal{X}, \xi^{\uparrow} + \epsilon^{\uparrow}, \xi^{\downarrow} + \epsilon^{\downarrow})$ for any $\epsilon^{\uparrow}, \epsilon^{\downarrow} > 0$. Thus, $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}} \geq 0$ and $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\downarrow}} \geq 0$, and we have the following lemma.

Lemma 3. *Nonnegativity of the derivatives: $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\uparrow}}$ and $\frac{\partial C(\mathcal{X}, \xi^{\uparrow}, \xi^{\downarrow})}{\partial \xi^{\downarrow}}$ are nonnegative.*

Since there are no fixed adjustment costs, any adjustment at a particular time can be viewed as the result of multiple simultaneous adjustments. Thus, starting with a smaller adjustment allows more flexibility and results in the nonnegativity of the derivatives.

5.2. The Set of Naturally Reachable States and Its Representation

Starting from an initial state \mathcal{X} , the state at time s will be $\sigma_s(\mathcal{X}) + w$ without any adjustment given a realization of the Brownian motion $W_s = w$. Thus, for any $s > 0$ and $w \in \mathbb{R}$, we call $\sigma_s(\mathcal{X}) + w$ a *naturally reachable state* from \mathcal{X} , and $\{\sigma_s(\mathcal{X}) + w : s > 0, w \in \mathbb{R}\} \subseteq \mathbb{D}$ is the set of all naturally reachable states from \mathcal{X} . For a fixed initial state \mathcal{X} , any naturally reachable state can be fully described by a pair $(w, s) \in \mathbb{R} \times \mathbb{R}_+$, referred to as a reachable state from a given initial state with a slight abuse of notation.

5.2.1. The Set of States Where No Adjustment Is Needed. At any naturally reachable state (w, s) from an initial state \mathcal{X} , an adjustment may or may not be needed. It is obvious that no upward (downward) adjustment should be made at \mathcal{X} if

$$\frac{\partial C(\mathcal{X}, 0, 0)}{\partial \xi^\uparrow} > 0 \left(\frac{\partial C(\mathcal{X}, 0, 0)}{\partial \xi^\downarrow} > 0 \right).$$

That is, the set of naturally reachable states in which no adjustment is needed is given by

$$\Xi_{\mathcal{X}} = \left\{ (w, s) \in \mathbb{R} \times \mathbb{R}_+ : \frac{\partial C(\sigma_s(\mathcal{X}) + w, 0, 0)}{\partial \xi^\uparrow} > 0, \frac{\partial C(\sigma_s(\mathcal{X}) + w, 0, 0)}{\partial \xi^\downarrow} > 0 \right\}. \tag{19}$$

Let

$$w_{\mathcal{X}}^\uparrow(s) = \max \left\{ w \in \mathbb{R} : \frac{\partial C(\sigma_s(\mathcal{X}) + w, 0, 0)}{\partial \xi^\uparrow} = 0 \right\}, \tag{20}$$

$$w_{\mathcal{X}}^\downarrow(s) = \min \left\{ w \in \mathbb{R} : \frac{\partial C(\sigma_s(\mathcal{X}) + w, 0, 0)}{\partial \xi^\downarrow} = 0 \right\}. \tag{21}$$

By Lemma 1,

$$\frac{\partial C(\sigma_s(\mathcal{X}) + w, 0, 0)}{\partial \xi^\uparrow} > 0 \left(\frac{\partial C(\sigma_s(\mathcal{X}) + w, 0, 0)}{\partial \xi^\downarrow} > 0 \right)$$

if and only if $w > w_{\mathcal{X}}^\uparrow(s)$ ($w < w_{\mathcal{X}}^\downarrow(s)$). Thus, (19) is equivalent to

$$\Xi_{\mathcal{X}} = \{ (w, s) \in \mathbb{R} \times \mathbb{R}_+ : w_{\mathcal{X}}^\uparrow(s) < w < w_{\mathcal{X}}^\downarrow(s) \}.$$

Since $\sigma_s(\mathcal{X})$ increases in s initially and remains constant when $s \geq \ell$, by Lemma 1, $w_{\mathcal{X}}^\uparrow(s)$ increases in s and stays constant at $w_0^\uparrow(\ell) + \mathcal{X}(\ell)$ for $s \geq \ell$, and $w_{\mathcal{X}}^\downarrow(s)$ decreases in s and stays constant at $w_0^\downarrow(\ell) + \mathcal{X}(\ell)$ for $s \geq \ell$ as shown in Figure 1. We do not give closed forms for the boundaries, but only shows the structural properties that the boundaries for each state \mathcal{X} are monotone. How to numerically solve the problem is another challenging research topic and beyond the scope of this paper.

At any given state \mathcal{X} , no adjustment is needed if $(0, 0) \in \Xi_{\mathcal{X}}$, or equivalently $w_{\mathcal{X}}^\downarrow(0) < 0 < w_{\mathcal{X}}^\uparrow(0)$. Otherwise, as ξ^\uparrow (ξ^\downarrow) increases, by Lemma 1, the marginal cost remains zero initially—that is, $\frac{\partial C(\mathcal{X}, \xi^\uparrow, \xi^\downarrow)}{\partial \xi^\uparrow}$ ($\frac{\partial C(\mathcal{X}, \xi^\uparrow, \xi^\downarrow)}{\partial \xi^\downarrow}$) stays at 0 for a while until it becomes positive. Since there are no fixed control costs, intuitively, the optimal upward (downward) adjustment should be obtained at the maximum ξ^\uparrow (ξ^\downarrow) at which the derivative is zero. This means that an upward (downward) adjustment is needed at time s if $w < w_{\mathcal{X}}^\uparrow(s)$ ($w > w_{\mathcal{X}}^\downarrow(s)$) as depicted in

Figure 1. $w_{\mathcal{X}}^\uparrow(s)$ and $w_{\mathcal{X}}^\downarrow(s)$ that define $\Xi_{\mathcal{X}}$.

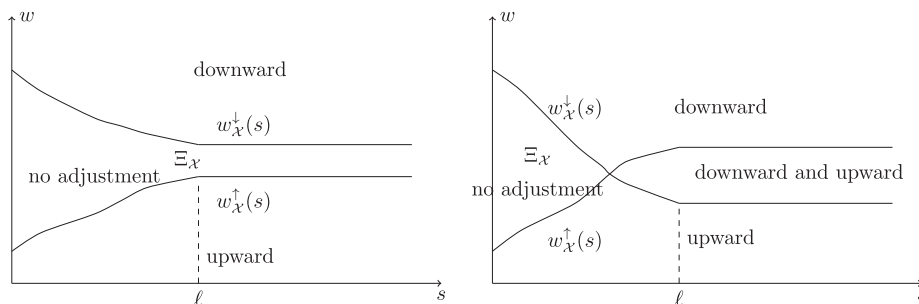


Figure 1, and simultaneous upward and downward adjustments are needed at a reachable state (w, s) if and only if $w_0^\dagger(\ell) \geq w_0^\dagger(\ell)$ as shown in the second case in Figure 1.

Furthermore, we show in the next proposition that $w_0^\dagger(\ell)$ and $w_0^\dagger(\ell)$ provide sufficient information for deciding whether an upward or downward adjustment is not needed at a state.

Proposition 4 (No Intervention Region). *No upward and downward adjustment is needed at \mathcal{X} if $\mathcal{X}(\ell) > w_0^\dagger(\ell)$ and $\mathcal{X}(\ell) < w_0^\dagger(\ell)$, respectively.*

Proof of Proposition 4. If $\mathcal{X}(\ell) < w_0^\dagger(\ell)$, we have $\mathcal{X} \leq w_0^\dagger(\ell)\mathcal{F}$. By part 3 of Lemma 1, $\frac{\partial C(\mathcal{X}, 0, 0)}{\partial \xi^\uparrow} \geq \frac{\partial C(w_0^\dagger(\ell)\mathcal{F}, 0, 0)}{\partial \xi^\uparrow} = 0$. Hence, by the definition of $w_0^\dagger(\ell)$, $\frac{\partial C(\mathcal{X}, 0, 0)}{\partial \xi^\uparrow} > 0$. If $\mathcal{X}(\ell) > w_0^\dagger(\ell)$, there exists $b > 0$ such that $\mathcal{X}(\ell) - b = w_0^\dagger(\ell)$ and $\mathcal{X} - b \leq w_0^\dagger(\ell)\mathcal{F}$. Then $\frac{\partial C(\mathcal{X}, 0, 0)}{\partial \xi^\uparrow} \geq \frac{\partial C(\mathcal{X} - b, 0, 0)}{\partial \xi^\uparrow} \geq \frac{\partial C(w_0^\dagger(\ell)\mathcal{F}, 0, 0)}{\partial \xi^\uparrow} = 0$ by parts 2 and 1 of Lemma 1, and $\frac{\partial C(\mathcal{X}, 0, 0)}{\partial \xi^\uparrow} > 0$ by the definition of $w_0^\dagger(\ell)$. \square

5.2.2. Properties of $V_{\mathcal{X}}(w, s)$ on $\Xi_{\mathcal{X}}$. In this section, we will show that the optimal value function $V_{\mathcal{X}}(w, s)$ defined in (11) is a solution to the HJB equation (13) with $\Xi_{\mathcal{X}}$ as the boundaries, and we will identify the timing of adjustment.

Theorem 3. *The partial derivatives $\frac{\partial V_{\mathcal{X}}(w, s)}{\partial s}$, $\frac{\partial V_{\mathcal{X}}(w, s)}{\partial w}$, and $\frac{\partial^2 V_{\mathcal{X}}(w, s)}{\partial w^2}$ exist, and*

$$\frac{\partial V_{\mathcal{X}}(w, s)}{\partial s} + \frac{\sigma^2}{2} \frac{\partial^2 V_{\mathcal{X}}(w, s)}{\partial w^2} + \mu \frac{\partial V_{\mathcal{X}}(w, s)}{\partial w} - \gamma V_{\mathcal{X}}(w, s) + h(\mathcal{X}(s) + w) = 0 \quad (22)$$

holds for almost every $(w, s) \in \Xi_{\mathcal{X}}$.

The key step in proving Theorem 3 is to establish the following property of $V_{\mathcal{X}}(w, s)$ in a small enough neighborhood of any point (\hat{w}, \hat{s}) in $\Xi_{\mathcal{X}}$.

Proposition 5 (Key Representation for the Value Function). *For a given $\mathcal{X} \in \mathbb{D}$ and $(\hat{w}, \hat{s}) \in \Xi_{\mathcal{X}}$, there exists a neighborhood of (\hat{w}, \hat{s}) , $\Xi_{\mathcal{X}}^{(\hat{w}, \hat{s})} \subset \Xi_{\mathcal{X}}$, such that*

$$V_{\mathcal{X}}(w, s) = \mathbb{E} \left[\int_0^{\tau} e^{-\gamma t} h(\mathcal{X}(s+t) + w + W_t) dt \right] + \mathbb{E} [e^{-\gamma \tau} V_{\mathcal{X}}(w + W_{\tau}, s + \tau)], \quad (23)$$

for all $(w, s) \in \Xi_{\mathcal{X}}^{(\hat{w}, \hat{s})}$, where τ is the first time the process $\{(w + W_t, s + t) : t \geq 0\}$ leaves $\Xi_{\mathcal{X}}^{(\hat{w}, \hat{s})}$.

Proof of Proposition 5. By the continuity of the partial derivatives in Lemma 2, there exist $\delta > 0$ and $k_0 > 0$ such that, for any $\hat{\mathcal{X}}$ satisfying $d(\sigma_{\hat{s}}(\hat{\mathcal{X}}) + \hat{w}, \hat{\mathcal{X}}) < 3\delta$,

$$\frac{\partial C(\hat{\mathcal{X}}, 0, 0)}{\partial \xi^\uparrow} \geq k_0, \quad \frac{\partial C(\hat{\mathcal{X}}, 0, 0)}{\partial \xi^\downarrow} \geq k_0. \quad (24)$$

Consider a neighborhood of (\hat{w}, \hat{s}) , $\Xi_{\mathcal{X}}^{(\hat{w}, \hat{s})} = (\hat{w}, \hat{s}) + B(\delta)$, where $B(\delta) := [-\delta, \delta] \times \left[0, \frac{\delta}{\gamma \hat{\mathcal{X}}(\delta)}\right]$. For any $(w, s) \in \Xi_{\mathcal{X}}^{(\hat{w}, \hat{s})}$, recall the definition of the distance $d(\cdot, \cdot)$ in Section 3.1,

$$\begin{aligned} d(\sigma_{\hat{s}}(\mathcal{X}) + \hat{w}, \sigma_s(\mathcal{X}) + w) &\leq d(\sigma_{\hat{s}}(\mathcal{X}) + \hat{w}, \sigma_s(\mathcal{X}) + \hat{w}) + d(\sigma_s(\mathcal{X}) + \hat{w}, \sigma_s(\mathcal{X}) + w) \\ &\leq (s - \hat{s})\gamma \hat{\mathcal{X}}(\ell) + \delta \leq 2\delta. \end{aligned}$$

Thus, $\Xi_{\mathcal{X}}^{(\hat{w}, \hat{s})} \subset \Xi_{\mathcal{X}}$. Next, show that the proposition holds in this neighborhood by contradiction via the following three steps.

1. Suppose (23) does not hold at a pair $(w', s') \in \Xi_{\mathcal{X}}^{(\hat{w}, \hat{s})}$. Since $V_{\mathcal{X}}(w', s') = C^*(\sigma_{s'}(\mathcal{X}) + w')$ is the optimal cost at $\sigma_{s'}(\mathcal{X}) + w'$, there exists a positive c_0 such that

$$\mathbb{E} \left[\int_0^{\tau'} e^{-\gamma t} h(\mathcal{X}(s'+t) + w' + W_t) dt \right] + \mathbb{E} [e^{-\gamma \tau'} V_{\mathcal{X}}(w' + W_{\tau'}, s' + \tau')] > V_{\mathcal{X}}(w', s') + c_0, \quad (25)$$

where τ' is the stopping time when $\{(w' + W_t, s' + t) : t \geq 0\}$ leaves $(\hat{w}, \hat{s}) + B(\delta)$. Introducing $\mathcal{X}' = \sigma_{s'}(\mathcal{X}) + w'$, (25) is equivalent to

$$\mathbb{E} \left[\int_0^{\tau'} e^{-\gamma t} h(\mathcal{X}'(t) + W_t) dt \right] + \mathbb{E} [e^{-\gamma \tau'} V_{\mathcal{X}'}(W_{\tau'}, \tau')] > V_{\mathcal{X}'}(0, 0) + c_0. \quad (26)$$

For any feasible periodic control policy $\pi^n = (Y^{n\uparrow}, Y^{n\downarrow})$ defined in (16), let \mathcal{X}'_t be the updated state at time t under this policy π^n . For any $\epsilon \leq \delta$, we define $N(\epsilon) = \inf\{k : \sum_{i=0}^k (\xi_i^{n\uparrow} + \xi_i^{n\downarrow}) \geq \epsilon\}$. Without loss of generality, assume that $\sum_{i=0}^{N(\epsilon)} (\xi_i^{n\uparrow} + \xi_i^{n\downarrow}) = \epsilon$; otherwise split the adjustments $\xi_{N(\epsilon)}^{n\uparrow}$ and/or $\xi_{N(\epsilon)}^{n\downarrow}$ into two. Let A be the event where the $N(\epsilon)$ th adjustment is made after τ' —that is, $A = \{\tau' \leq T_{N(\epsilon)}^n\}$.

2. Estimate the cost $C(\mathcal{X}', \pi^n)$ by considering the events A and A^c , respectively.

a. On the event A , from (24), the marginal costs for the upward and downward adjustments are quite large, implying that

$$C(\mathcal{X}', \pi^n) \geq C^*(\mathcal{X}') + \mathbb{P}(A)e^{-\gamma\delta}k_0\epsilon \geq V_{\mathcal{X}'}(0, 0) + \mathbb{P}(A)e^{-\gamma\delta}k_0\epsilon. \tag{27}$$

b. On the event A^c , the cumulative amount of upward and downward adjustment by the stopping time τ' is less than ϵ —that is, $d(\mathcal{X}'_s, \sigma_s(\mathcal{X}')) + W_s < \epsilon$ for any $0 \leq s \leq \tau'$. Combining with (26), we can imply that

$$C(\mathcal{X}', \pi^n) \geq V_{\mathcal{X}'}(0, 0) + c_0 - \frac{M\epsilon}{\gamma} - \mathbb{P}(A)(\delta\bar{h} + \bar{V}), \tag{28}$$

where \bar{h} and \bar{V} are two constants. The detailed proofs of (27) and (28) are presented in the appendix.

3. Properly choose $\epsilon = \min\{\delta, \frac{c_0\gamma}{2M}\}$ and denote $p_0 = \frac{c_0}{2e^{-\gamma\delta}k_0 \min\{\delta, \frac{c_0\gamma}{2M}\} + 2\delta\bar{h} + 2\bar{V}}$. If $\mathbb{P}(A) \geq p_0$, from (27) $C(\mathcal{X}', \pi) \geq V_{\mathcal{X}'}(0, 0) + e^{-\gamma\delta}k_0 \min\{\delta, \frac{c_0\gamma}{2M}\}p_0$. Otherwise, if $\mathbb{P}(A) \leq p_0$, from (28) $C(\mathcal{X}', \pi) \geq V_{\mathcal{X}'}(0, 0) + e^{-\gamma\delta}k_0 \min\{\delta, \frac{c_0\gamma}{2M}\}p_0$. Thus, for any discrete policy π^n , its associated expected cost will be at least $V_{\mathcal{X}'}(0, 0) + e^{-\gamma\delta}k_0 \min\{\delta, \frac{c_0\gamma}{2M}\}p_0$. However, this is a contradiction of Proposition 3. \square

We are now ready to prove Theorem 3.

Proof of Theorem 3. In Proposition 5, we have proved that for any $(\hat{w}, \hat{s}) \in \Xi_{\mathcal{X}}$, we can find a corresponding subset $\Xi_{\mathcal{X}}^{(\hat{w}, \hat{s})} \subseteq \Xi_{\mathcal{X}}$ such that all of the points in the subset satisfy (23). Applying Dynkin’s law in Dynkin [8] to (23), we find that (22) holds for all (w, s) in $\Xi_{\mathcal{X}}^{(\hat{w}, \hat{s})}$. In other words, for any $(\hat{w}, \hat{s}) \in \Xi_{\mathcal{X}}$, we can find a corresponding neighborhood $\Xi_{\mathcal{X}}^{(\hat{w}, \hat{s})} \in \Xi_{\mathcal{X}}$ where (22) holds. Since $\Xi_{\mathcal{X}} = \bigcup_{(\hat{w}, \hat{s}) \in \Xi_{\mathcal{X}}} \Xi_{\mathcal{X}}^{(\hat{w}, \hat{s})}$, we can conclude that (22) holds for all points in $\Xi_{\mathcal{X}}$. \square

Define $\tau_{\mathcal{X}} \geq 0$ to be the first time the process $\{(w + W_t, s + t) : t \geq 0\}$ leaves $\Xi_{\mathcal{X}}$. By Ito’s formula and Theorem 3, we have the following corollary (whose proof is skipped as it is the same as that of Theorem 3). The corollary helps to identify the time of the adjustment since the equation in the corollary actually holds for any stopping time τ such that $\tau \leq \tau_{\mathcal{X}}$ with probability 1.

Corollary 1. For any given $\mathcal{X} \in \mathbb{D}$ and $(w, s) \in \Xi_{\mathcal{X}}$,

$$V_{\mathcal{X}}(w, s) = \mathbb{E} \left[\int_0^{\tau_{\mathcal{X}}} e^{-\gamma t} h(\mathcal{X}(s+t) + w + W_t) dt \right] + \mathbb{E} [e^{-\gamma\tau_{\mathcal{X}}} V_{\mathcal{X}}(w + W_{\tau_{\mathcal{X}}}, s + \tau_{\mathcal{X}})].$$

Based on the L^1 -convexity of $C^*(\mathcal{X})$ and its optimality, we have the following proposition.

Proposition 6. The partial derivatives $\frac{\partial V_{\mathcal{X}}(w, s)}{\partial s}$, $\frac{\partial V_{\mathcal{X}}(w, s)}{\partial w}$, and $\frac{\partial^2 V_{\mathcal{X}}(w, s)}{\partial w^2}$ exist, and

$$\frac{\partial V_{\mathcal{X}}(w, s)}{\partial s} + \frac{\sigma^2}{2} \frac{\partial^2 V_{\mathcal{X}}(w, s)}{\partial w^2} + \mu \frac{\partial V_{\mathcal{X}}(w, s)}{\partial w} - \gamma V_{\mathcal{X}}(w, s) + h(\mathcal{X}(s) + w) \geq 0$$

holds for almost every $(w, s) \in \mathbb{R} \times \mathbb{R}_+$.

The above proposition and Lemma 3 show that each one of the three terms of (13) is always nonnegative. Moreover, if $\mathcal{X} \in \Xi_{\mathcal{X}}$, the first term of (13) must be zero by Theorem 3. Otherwise, by the definition of $\Xi_{\mathcal{X}}$, at least one of the last two terms of (13) is zero. This yields the following theorem.

Theorem 4. For any $\mathcal{X} \in \mathbb{D}$, $V_{\mathcal{X}}(w, s)$ is a solution to the HJB equations (13) with $\Xi_{\mathcal{X}}$ as the boundaries.

6. The Optimal Control Policy

In this section, we will construct an optimal control policy. We will first define the set of states in which an upward or downward adjustment is needed. We then examine the corresponding upward (downward) adjustment policy for a given downward (upward) adjustment policy, referred to as the one-sided reflection

mapping in Section 6.1.1, and construct a two-sided reflection mapping in Section 6.1.2. Lastly, in Section 6.2, we show that the two-sided reflection mapping is an optimal control.

Recall that $\mathbb{E}_{\mathcal{X}}$ is divided into three regions as illustrated in Figure 1. Proposition 5 states that no upward or downward control is needed in the middle region, and Equations (20)–(21) reveal that the timing of exercising upward (downward) control is when the state \mathcal{X} hits the lower (upper) boundary—that is, at time s when $w < w_{\mathcal{X}}^{\uparrow}(s)$ ($w > w_{\mathcal{X}}^{\downarrow}(s)$). Thus, the optimal policy should exert the minimum amounts of control sufficient to keep the state \mathcal{X}_t out of \mathbb{D}^{\uparrow} and \mathbb{D}^{\downarrow} , where

$$\mathbb{D}^{\uparrow} = \{\mathcal{X} \in \mathbb{D} : w_{\mathcal{X}}^{\uparrow}(0) < 0\} \text{ and } \mathbb{D}^{\downarrow} = \{\mathcal{X} \in \mathbb{D} : w_{\mathcal{X}}^{\downarrow}(0) > 0\}, \quad (29)$$

which is the two-sided reflection mapping. Lemma 1 immediately leads to the following corollary.

Corollary 2. *Let $\bar{\mathbb{D}}^{\uparrow}$ and $\bar{\mathbb{D}}^{\downarrow}$ be the complements of \mathbb{D}^{\uparrow} and \mathbb{D}^{\downarrow} , respectively.*

1. *If $\mathcal{X} \geq \mathcal{X}'$ and $\mathcal{X}(\ell) = \mathcal{X}'(\ell)$, then $\mathcal{X} \in \bar{\mathbb{D}}^{\uparrow}$ implies $\mathcal{X}' \in \bar{\mathbb{D}}^{\uparrow}$.*
2. *If $\mathcal{X} \in \bar{\mathbb{D}}^{\uparrow}$, then $\mathcal{X} + a \in \bar{\mathbb{D}}^{\uparrow}$ for all $a > 0$.*
3. *If $\mathcal{X} \geq \mathcal{X}'$, then $\mathcal{X} \in \bar{\mathbb{D}}^{\downarrow}$ implies $\mathcal{X}' \in \bar{\mathbb{D}}^{\downarrow}$.*

If there exists a state belonging to both \mathbb{D}^{\uparrow} and \mathbb{D}^{\downarrow} , then we need to make downward and upward adjustments at the same time. This can happen when it is too costly to hold a unit of inventory that is likely to be needed ℓ amount of time later—that is, when the cost for holding a large amount of inventory is relatively high and the lead time is relatively long. When this happens, the optimal adjustment can be quite complicated. Thus, we will focus on the case where $\mathbb{D}^{\uparrow} \cap \mathbb{D}^{\downarrow} = \emptyset$, which holds in most real applications. The following lemma also provides an explicit sufficient condition for this to hold.

Lemma 4. *$\mathbb{D}^{\uparrow} \cap \mathbb{D}^{\downarrow} = \emptyset$ if and only if $w_0^{\downarrow}(\ell) > w_0^{\uparrow}(\ell)$. A sufficient condition for $\mathbb{D}^{\uparrow} \cap \mathbb{D}^{\downarrow} = \emptyset$ is $k^{\uparrow} + k^{\downarrow} > \frac{1-e^{-\gamma\ell}}{\gamma} \max_{x>0} h'(x)$.*

Proof of Lemma 4. A direct result from Figure 1 is that a necessary and sufficient condition for nonsimultaneous upward and downward adjustments is $w_0^{\downarrow}(\ell) > w_0^{\uparrow}(\ell)$. If these two subsets intersect and $(\xi^{\uparrow}, \xi^{\downarrow})$ are simultaneously adjusted, for a downward and an upward adjustment $(\xi^{\uparrow} - \epsilon, \xi^{\downarrow} - \epsilon)$, we increase the holding cost by no more than $\epsilon \int_0^{\ell} e^{-\gamma t} \max_{x>0} h'(x) dt$ while reducing the control cost by $(k^{\uparrow} + k^{\downarrow})\epsilon$. Since $k^{\uparrow} + k^{\downarrow} > \max_{x>0} h'(x) \frac{1-e^{-\gamma\ell}}{\gamma}$, the total cost will decrease. \square

6.1. Reflection Mappings

We first identify the minimum upward (downward) adjustment needed to ensure $\mathcal{X}_s \in \bar{\mathbb{D}}^{\uparrow}$ ($\mathcal{X}_s \in \bar{\mathbb{D}}^{\downarrow}$) at all $s \geq 0$ for a given downward (upward) adjustment. We refer to them as one-sided reflection mappings that will lead to the two-sided reflection policy, an optimal control.

6.1.1. One-Sided Reflection Mappings. For a given sample path of the Brownian motion ω and initial state \mathcal{X} , the state \mathcal{X}_s under policy $(Y^{\uparrow}, Y^{\downarrow})$ can also be written as

$$\mathcal{X}_s = \sigma_s(\mathcal{X}) - Y^{\downarrow}(s) + \sigma_{s-\ell}(Y^{\uparrow}) \wedge Y^{\uparrow}(s) \mathcal{F} + \omega(s)$$

by the dynamics (2). For convenience, we use the superscripts $i, j \in \{\uparrow, \downarrow\}$, $i \neq j$, to indicate a pair of upward and downward adjustments. For any given $(\mathcal{X}, Y^j, \omega)$,

$$\Pi^i(\mathcal{X}, Y^j, \omega) = \{Y^i : \mathcal{X}_s = \sigma_s(\mathcal{X}) - Y^{\downarrow}(s) + \sigma_{s-\ell}(Y^{\uparrow}) \wedge Y^{\uparrow}(s) \mathcal{F} + \omega(s) \in \bar{\mathbb{D}}^i, \text{ for all } s \geq 0\} \quad (30)$$

is the set of all of the feasible one-sided adjustments Y^i that will ensure $\mathcal{X}_s \in \bar{\mathbb{D}}^i$ at all s . Recall that \mathbb{D} is a functional set. For any subset $\emptyset \neq \mathbb{S} \subseteq \mathbb{D}$, let $\inf \mathbb{S}$ be a function that takes the infimum of all functions in \mathbb{S} at any point—that is,

$$(\inf \mathbb{S})(t) = \inf_{f \in \mathbb{S}} \{f(t)\} \quad \text{for any } t \geq 0.$$

By lemma 14.2.2 in Whitt [23], $\inf \mathbb{S} \in \mathbb{D}$.

Definition 4 (One-sided reflection mappings). We call $\psi^i : (\mathbb{D}, \mathbb{D}, \mathbb{D}) \rightarrow \mathbb{D}$ a one-sided reflection mapping for \mathbb{D}^i if, for a given state \mathcal{X} , sample path ω and $Y^j \in \mathbb{D}$,

$$\psi^i(\mathcal{X}, Y^j, \omega) = \inf \Pi^i(\mathcal{X}, Y^j, \omega).$$

Next, we show the existence of the one-sided reflection mappings in Proposition 7 and provide some properties of the mappings in Proposition 8.

Proposition 7 (Existence of the Reflection Maps). $\psi^i(\mathcal{X}, Y^i, \omega)$ exists and belongs to $\Pi^i(\mathcal{X}, Y^i, \omega)$.

Proof of Proposition 7. Since the proofs are similar, we only prove the result for $\psi^\uparrow(\mathcal{X}, Y^\downarrow, \omega)$. We first claim that $\Pi^\uparrow(\mathcal{X}, Y^\downarrow, \omega)$ is nonempty as an adjustment

$$g(t) = \sup_{0 \leq u \leq t} \{w_0^\uparrow(t) - \omega(u) + Y^\downarrow(u) - \mathcal{X}(u + t)\} \in \Pi^\uparrow(\mathcal{X}, Y^\downarrow, \omega).$$

This is because

$$\mathcal{X}_s(t) = \mathcal{X}(s + t) + \omega(s) - Y^\downarrow(s) + g(s) \geq w_0^\uparrow(t), \text{ for any } s \geq 0$$

and by Proposition 4, $\frac{\partial C(\mathcal{X}_s, 0, 0)}{\partial \xi^\uparrow} = 0$. So $\Pi^\uparrow(\mathcal{X}, Y^\downarrow, \omega)$ at least has one element.

It remains to be shown that $\psi^\uparrow(\mathcal{X}, Y^\downarrow, \omega) \in \Pi^\uparrow(\mathcal{X}, Y^\downarrow, \omega)$. For any fixed $\epsilon > 0$ and $s \geq 0$, there exists $Y^{\uparrow'} \in \Pi^\uparrow(\mathcal{X}, Y^\downarrow, \omega)$ such that $Y^{\uparrow'} \geq \psi^\uparrow(\mathcal{X}, Y^\downarrow, \omega)$ and $Y^{\uparrow'}(s) \leq \psi^\uparrow(s) + \epsilon$. Thus, $\sigma_s(\mathcal{X}) + w(s) - Y^\downarrow(s) + \sigma_{s-\ell}(Y^{\uparrow'}) \wedge Y^{\uparrow'}(s) \mathcal{F} \in \bar{\mathbb{D}}^\uparrow$. Then, by parts 1 and 2 of Corollary 2, we know that $\sigma_s(\mathcal{X}) + w(s) - Y^\downarrow(s) + \sigma_{s-\ell}(\psi^\uparrow(s) + \epsilon) \wedge (\psi^\uparrow(s) + \epsilon) \mathcal{F} \in \bar{\mathbb{D}}^\uparrow$. Because s and ϵ are arbitrary, $\psi^\uparrow(\mathcal{X}, Y^\downarrow, \omega) \in \Pi^\uparrow(\mathcal{X}, Y^\downarrow, \omega)$. \square

Proposition 8. Let \mathcal{X} be the initial state and ω a sample path.

- $\psi^\uparrow(\mathcal{X}, Y^\downarrow, \omega)$ decreases in Y^\downarrow and $\psi^\downarrow(\mathcal{X}, Y^\uparrow, \omega)$ increases in Y^\uparrow .
- $\sup_{0 \leq u \leq t} |\psi^i(\mathcal{X}, Y_1^i, \omega)(u) - \psi^i(\mathcal{X}, Y_2^i, \omega)(u)| \leq \sup_{0 \leq u \leq t} |Y_1^i(u) - Y_2^i(u)|$ for any given $t \geq 0$, hence $\psi^i(\mathcal{X}, Y^i, \omega)$ is Lipschitz continuous in Y^i under the uniform norm.

Proof of Proposition 8. We will only prove the results for $\psi^\uparrow(\mathcal{X}, Y^\downarrow, \omega)$.

1. Suppose $Y_1^\downarrow \geq Y_2^\downarrow$. For any $Y \in \Pi^\uparrow(\mathcal{X}, Y_1^\downarrow, \omega)$, $\sigma_s(\mathcal{X}) + \sigma_{(s-\ell)}(Y) \wedge Y(s) \mathcal{F} - Y_1^\downarrow(s) + \omega(s) \in \bar{\mathbb{D}}^\uparrow$. By part 2 of Corollary 2, $\sigma_s(\mathcal{X}) + \sigma_{(s-\ell)}(Y) \wedge Y(s) \mathcal{F} - Y_2^\downarrow(s) + \omega(s) \in \bar{\mathbb{D}}^\uparrow$ for all $s \geq 0$ and consequently $Y \in \Pi^\uparrow(\mathcal{X}, Y_2^\downarrow, \omega)$. Thus, $\Pi(\mathcal{X}, Y_1^\downarrow, \omega) \subseteq \Pi(\mathcal{X}, Y_2^\downarrow, \omega)$ and $\psi^\uparrow(\mathcal{X}, Y_1^\downarrow, \omega) \leq \psi^\uparrow(\mathcal{X}, Y_2^\downarrow, \omega)$.

2. We prove this part by contradiction. For convenience, let $a_0 = \sup_{0 \leq u \leq t} |Y_1^\downarrow(u) - Y_2^\downarrow(u)| < \infty$ and $g_1 = \psi^\uparrow(\mathcal{X}, Y_1^\downarrow, \omega)$ and $g_2 = \psi^\uparrow(\mathcal{X}, Y_2^\downarrow, \omega)$. Suppose that the inequality does not hold. Define $\tau := \inf \{s \geq 0 : |g_2(s) - g_1(s)| > a_0\}$. Without loss of generality, we assume $g_2(\tau) \geq g_1(\tau) + a_0$. Because g_1 and g_2 are right-continuous, there exists an $\epsilon < \ell$ such that $g_2(s) - g_1(s) > a_0$ for $s \in (\tau, \tau + \epsilon]$. Consider the following function

$$g'_2(u) = \begin{cases} g_1(u) + a_0 & u \in [\tau, \tau + \epsilon), \\ g_2(u) & \text{otherwise.} \end{cases}$$

Then, for all $t < \tau$, $g'_2(t) = g_2(t) \leq g_1(t) + a_0 = g'_2(t)$ and $g'_2(\tau + \epsilon) = g_2(\tau + \epsilon) > g_1(\tau + \epsilon) + a_0$. Thus, g'_2 is also nondecreasing and strictly less than g_2 . Next, we show that $g'_2 \in \Pi(\mathcal{X}, Y_2^\downarrow, \omega)$ or equivalently, for all $s \geq 0$,

$$\sigma_s(\mathcal{X}) + \sigma_{(s-\ell)} g'_2 \wedge g'_2(s) \mathcal{F} - Y_2^\downarrow(s) + \omega(s) \in \bar{\mathbb{D}}^\uparrow \tag{33}$$

and hence, we have a contradiction. Note that, $\sigma_s(\mathcal{X}) + \sigma_{(s-\ell)} g_k \wedge g_k(s) \mathcal{F} - Y_k^\downarrow(s) + \omega(s) \in \bar{\mathbb{D}}^\uparrow$ for $k = 1, 2$.

- For $0 \leq s < \tau$, $\sigma_{(s-\ell)}(g'_2) \wedge g'_2(s) \mathcal{F} = \sigma_{(s-\ell)}(g_2) \wedge g_2(s) \mathcal{F}$, and (33) holds.
- For $\tau \leq s < \tau + \epsilon$, $g'_2(s) = g_1(s) + a_0$ and $\sigma_{(s-\ell)}(g'_2) \wedge g'_2(s) \mathcal{F} + a_0 \geq \sigma_{(s-\ell)}(g_1) \wedge g_1(s) \mathcal{F}$. By part 1 of Corollary 2, $\sigma_s(\mathcal{X}) + \sigma_{(s-\ell)}(g'_2) \wedge g'_2(s) \mathcal{F} - a_0 - Y_1^\downarrow(s) + \omega(s) \in \bar{\mathbb{D}}^\uparrow$. Since $a_0 + Y_1^\downarrow(s) \geq Y_2^\downarrow(s)$, (33) holds by part 2 of Corollary 2.
- For $s > \tau + \epsilon$, $g'_2(s) = g_2(s)$ and $\sigma_{(s-\ell)}(g'_2) \wedge g'_2(s) \mathcal{F} \leq \sigma_{(s-\ell)}(g_2) \wedge g_2(s) \mathcal{F}$. By part 1 of Corollary 2, (33) holds. \square

Because of the “inf” operator, $\psi^i(\mathcal{X}, Y^i, \omega)(t)$ increases in t only when \mathcal{X}_t hits the boundary of \mathbb{D}^i —that is, $\frac{\partial C(\mathcal{X}_t, 0, 0)}{\partial \xi^i} = 0$, which is summarized in the following proposition.

Proposition 9 (Complementarity of the Reflection Mappings). If \mathcal{X}_t is the state at time t under policy ψ^i for a given Y^i and initial state \mathcal{X} , then $\int_a^b \frac{\partial C(\mathcal{X}_t, 0, 0)}{\partial \xi^i} d\psi^i(\mathcal{X}, Y^i, \omega)(t) = 0$ for any $0 \leq a \leq b \leq \infty$.

6.1.2. A Two-Sided Reflection Mapping. We are now ready to define a two-sided reflection mapping, and show its existence and uniqueness.

Definition 5 (A Two-Sided Reflection Mapping). For a given \mathcal{X} and Brownian motion sample path ω , $(Y^\uparrow, Y^\downarrow)$ is called a two-sided reflection mapping if

$$Y^\uparrow = \psi^\uparrow(\mathcal{X}, Y^\downarrow, \omega), \tag{34}$$

$$Y^\downarrow = \psi^\downarrow(\mathcal{X}, Y^\uparrow, \omega). \tag{35}$$

Proposition 10. For any given \mathcal{X} and Brownian motion sample path ω , there exists a unique two-sided reflection mapping $(Y^{\uparrow*}, Y^{\downarrow*})$.

Proof of Proposition 10. The existence of a two-sided mapping: We show the existence of a two-sided mapping as the limit of a series of one-sided mappings and the convergence of the mappings is achieved in a finite number of steps. For any \mathcal{X} and sample path ω , we construct a series of upward and downward reflection mappings as $Y_0^\downarrow = 0$ and

$$Y_k^\uparrow = \psi^\uparrow(\mathcal{X}, Y_{k-1}^\downarrow, \omega), \tag{36}$$

$$Y_k^\downarrow = \psi^\downarrow(\mathcal{X}, Y_k^\uparrow, \omega), \tag{37}$$

for $k = 1, 2, 3, \dots$. By part 1 of Proposition 8, one can easily see that both Y_k^\uparrow and Y_k^\downarrow increase in k (in the sense of “ \leq ”) and hence converge as $k \rightarrow \infty$. We now show that, for any fixed t , both $Y_k^\uparrow(t)$ and $Y_k^\downarrow(t)$ converge in a finite number of steps.

Let $\mathcal{X}_s^{\uparrow k} \in \mathbb{D}^\uparrow$ denote the resulting state at time s under policy $(Y_k^\uparrow, Y_{k-1}^\downarrow)$ and $\mathcal{X}_s^{\downarrow k} \in \mathbb{D}^\downarrow$ denote the state at time s under policy $(Y_k^\downarrow, Y_k^\uparrow)$, for $k = 1, 2, \dots$. Let $t_k^\uparrow = \inf\{t : \mathcal{X}_t^{\uparrow k} \in \mathbb{D}^\uparrow\}$ and $t_k^\downarrow = \inf\{t : \mathcal{X}_t^{\downarrow k} \in \mathbb{D}^\downarrow\}$ be the first time $\mathcal{X}_t^{\uparrow k}$ enters \mathbb{D}^\downarrow and $\mathcal{X}_t^{\downarrow k}$ enters \mathbb{D}^\uparrow , respectively.

We first prove that for any given $k \geq 1$, $Y_m^\downarrow = Y_{k-1}^\downarrow$ on $[0, t_k^\uparrow]$ for all $m \geq k$. The proof of $Y_m^\uparrow = Y_k^\uparrow$ on $[0, t_k^\downarrow]$ for all $m \geq k$ is similar and hence omitted. Thus, Y_m^\uparrow and Y_m^\downarrow converge to Y_k^\uparrow and Y_k^\downarrow in k steps on $[0, t_{k+1}^\uparrow]$ and $[0, t_k^\downarrow]$, respectively. Since $Y_{k-1}^\downarrow(s) \leq Y_k^\downarrow(s) = \psi^\downarrow(\mathcal{X}, Y_k^\uparrow, \omega)(s)$ for all $s \geq 0$, Y_{k-1}^\downarrow is a smaller downward adjustment than Y_k^\downarrow and can also prevent the profile from entering \mathbb{D}^\downarrow as $\mathcal{X}_s^{\uparrow k} \in \mathbb{D}^\downarrow$ for $s \in [0, t_k^\uparrow]$. Note that the one-sided mappings (36) and (37) on $[0, s]$ only depend on the sample path ω on $[0, s]$. Thus, $Y_{k-1}^\downarrow = Y_k^\downarrow$ on $[0, t_k^\uparrow]$ implying that $(Y_k^\uparrow, Y_{k-1}^\downarrow)$ jointly satisfy (34) and (35) on $[0, t_k^\uparrow]$. Hence, $Y_m^\uparrow = Y_k^\uparrow$ and $Y_m^\downarrow = Y_{k-1}^\downarrow$ on $[0, t_k^\uparrow]$ for $m \geq k$.

Next, we show that $t_k^\uparrow \leq t_k^\downarrow \leq t_{k+1}^\uparrow \leq t_{k+1}^\downarrow$ for any given $k \geq 1$. Since $Y_k^\downarrow = Y_{k-1}^\downarrow$ on $[0, t_k^\uparrow]$, $\mathcal{X}_s^{\downarrow k} = \mathcal{X}_s^{\downarrow k-1}$ for $s \in [0, t_k^\uparrow]$ and $t_k^\uparrow \leq t_k^\downarrow$. Likewise, since $Y_{k+1}^\uparrow = Y_k^\uparrow$ on $[0, t_k^\downarrow]$, $t_k^\downarrow \leq t_{k+1}^\uparrow$.

It remains for us to show that, for any fixed t , there exists k' such that $t_k^\downarrow \geq t$. Denote $s_k = \inf\{t_k^\downarrow \leq t \leq t_k^\uparrow : Y_k^\downarrow(t) = Y_k^\downarrow(t_k^\downarrow)\}$. Then, no adjustment is made on $[s_k, t_k^\downarrow]$, and $\mathcal{X}_t^{\downarrow k}$ enters \mathbb{D}^\uparrow after t_k^\downarrow because of the Brownian motion ω . If $Y_k^\downarrow(s_k) > Y_k^\downarrow(s_k^-)$, by Proposition 9, $\frac{\partial C(\mathcal{X}_{s_k}^{\downarrow k}, 0, 0)}{\partial \xi^1} = 0$. On the other hand, if $Y_k^\downarrow(s_k) = Y_k^\downarrow(s_k^-)$, we can find an increasing sequence $\{u_p, p = 1, 2, \dots\}$ such that $\lim_{p \rightarrow \infty} u_p = s_k$ and Y_k^\downarrow increases at u_p . By Proposition 9, we have $\frac{\partial C(\mathcal{X}_{u_p}^{\downarrow k}, 0, 0)}{\partial \xi^1} = 0$ for $p = 1, 2, \dots$, which implies that $\frac{\partial C(\mathcal{X}_{s_k}^{\downarrow k}, 0, 0)}{\partial \xi^1} = 0$ following the continuity property in Lemma 2. Then, by Proposition 4, we must have $\mathcal{X}(s_k) + \omega(s_k) - Y_k^\downarrow(s_k) + Y_k^\uparrow(s_k) \geq w_0^\downarrow(\ell)$ and $\mathcal{X}(t_k^\downarrow) + \omega(t_k^\downarrow) - Y_k^\downarrow(t_k^\downarrow) + Y_k^\uparrow(t_k^\downarrow) \leq w_0^\downarrow(\ell)$.

Since $Y_k^\downarrow(s_k) = Y_k^\downarrow(t_k^\downarrow)$, $\mathcal{X}(s_k) \leq \mathcal{X}(t_k^\downarrow)$ and $Y_k^\uparrow(s_k) \leq Y_k^\uparrow(t_k^\downarrow)$, we have

$$\omega(s_k) - \omega(t_k^\downarrow) \geq w_0^\downarrow(\ell) - w_0^\downarrow(\ell). \tag{38}$$

By the continuity of the sample path ω , there exists a $\delta > 0$ such that

$$\sup_{\substack{|u_1 - u_2| < \delta \\ 0 \leq u_1 < u_2 \leq t}} |\omega(u_1) - \omega(u_2)| < \frac{w_0^\downarrow(\ell) - w_0^\downarrow(\ell)}{2}.$$

This implies that $t_k^\downarrow \geq s_k + \delta \geq t_k^\uparrow + \delta \geq t_{k-1}^\downarrow + \delta$ if $t_k^\downarrow < t$. So $t_{\lfloor \frac{t}{\delta} \rfloor}^\downarrow \geq t$.

Let $(Y^{\uparrow*}, Y^{\downarrow*})$ be the pointwise limit of the sequence $\{(Y_k^\uparrow, Y_k^\downarrow) : k = 1, 2, \dots\}$. Since convergence can be achieved in a finite number of steps for any given t , $(Y^{\uparrow*}, Y^{\downarrow*})$ are finite at all $t \geq 0$. Taking the limit on both sides of (36) and (37) by the Lipschitz continuity of $\psi^\uparrow(\mathcal{X}, Y^\downarrow, \omega)$ and $\psi^\downarrow(\mathcal{X}, Y^\uparrow, \omega)$, we can show that $(Y^{\uparrow*}, Y^{\downarrow*})$ jointly satisfy (34) and (35).

The uniqueness of the two-sided mapping: Finally, we prove the uniqueness of the two-sided mapping. Suppose that there exists a two-sided mapping $(Y^{\uparrow'}, Y^{\downarrow'})$ that satisfies (34) and (35). By part 1 of Proposition 8, $Y^{\downarrow'} \geq 0$ implies $Y^{\uparrow'} \geq Y_1^\uparrow$ and $Y^{\downarrow'} \geq Y_1^\downarrow$, and subsequently, $Y^{\uparrow'} \geq Y_i^\uparrow$ and $Y^{\downarrow'} \geq Y_i^\downarrow$ for $i = 2, 3, \dots$. Thus, $Y^{\uparrow'} \geq Y^{\uparrow*}$ and $Y^{\downarrow'} \geq Y^{\downarrow*}$. Define $\tau^\uparrow = \inf\{t \geq 0 : Y^{\uparrow'}(t) > Y^{\uparrow*}(t)\}$ and $\tau^\downarrow = \inf\{t \geq 0 : Y^{\downarrow'}(t) > Y^{\downarrow*}(t)\}$.

1. If $\tau^\uparrow > \tau^\downarrow$, then $Y^{\downarrow'}(u) = Y^{\downarrow*}(u)$ for $\tau^\downarrow \leq u < \tau^\uparrow$. Let

$$Y^{\uparrow''}(u) = \begin{cases} Y^\uparrow(u), & u \in [0, \tau^\uparrow), \\ Y^{\uparrow'}(u), & \text{otherwise.} \end{cases}$$

Since $Y^{\uparrow''}(u) = Y^\uparrow(u)$ for $u < \tau^\uparrow$, $Y^{\uparrow''}$ is increasing and strictly less than $Y^{\uparrow'}$. By part 2 of Corollary 2, $Y^{\uparrow''} \in \Pi^\uparrow(\mathcal{X}, Y^{\downarrow'}, \omega)$, a contradiction.

2. If $\tau^\downarrow > \tau^\uparrow$, the proof is similar and omitted.

3. If $\tau^\uparrow = \tau^\downarrow$, there exists some $\delta' > 0$ such that both $Y^{\uparrow'} - Y^\uparrow$ and $Y^{\downarrow'} - Y^\downarrow$ are strictly positive in $(\tau^\uparrow, \tau^\uparrow + \delta)$. Denote $A_0 = \mathcal{X}(\tau^\uparrow + \ell) + \omega(\tau^\uparrow) + Y^{\uparrow'}(\tau^\uparrow) - Y^{\downarrow'}(\tau^\uparrow)$.

• $A_0 \geq \frac{w_0^\downarrow(\ell) + w_0^\uparrow(\ell)}{2}$: Since $\mathcal{X}(t)$, $\omega(t)$, and $Y^{\downarrow'}$ are right-continuous, there exists a $\delta \leq \delta'$ such that $|\mathcal{X}(t + \ell) - \mathcal{X}(\tau^\uparrow + \ell)| < \frac{w_0^\downarrow(\ell) - w_0^\uparrow(\ell)}{8}$, $|Y^{\downarrow'}(t) - Y^{\downarrow'}(\tau^\uparrow)| < \frac{w_0^\downarrow(\ell) - w_0^\uparrow(\ell)}{8}$, and $|\omega(t) - \omega(\tau^\uparrow)| < \frac{w_0^\downarrow(\ell) - w_0^\uparrow(\ell)}{8}$ when $t \in [\tau^\uparrow, \tau^\uparrow + \delta)$.

Choose $\epsilon < \frac{w_0^\downarrow(\ell) - w_0^\uparrow(\ell)}{8}$ and let

$$Y^{\uparrow''}(u) = \begin{cases} \begin{cases} Y^{\uparrow'}(\tau^\uparrow) - \epsilon & \text{if } Y^{\uparrow'}(\tau^\uparrow) > Y^\uparrow(\tau^\uparrow), \\ Y^{\uparrow'}(\tau^\uparrow) & \text{if } Y^{\uparrow'}(\tau^\uparrow) = Y^\uparrow(\tau^\uparrow), \end{cases} & u \in [\tau^\uparrow, \tau^\uparrow + \delta), \\ Y^{\uparrow'} & \text{otherwise.} \end{cases}$$

Then, under adjustments $(Y^{\uparrow''}, Y^{\downarrow'})$,

$$\begin{aligned} \mathcal{X}_t(\ell) &= \mathcal{X}(t + \ell) + \omega(t) + Y^{\uparrow''}(t) - Y^{\downarrow'}(t) \\ &\geq \mathcal{X}(\tau^\uparrow + \ell) + \omega(\tau^\uparrow) + Y^{\uparrow'}(\tau^\uparrow) - Y^{\downarrow'}(\tau^\uparrow) - \frac{w_0^\downarrow(\ell) - w_0^\uparrow(\ell)}{2} \geq A_0 - \frac{w_0^\downarrow(\ell) - w_0^\uparrow(\ell)}{2} > w_0^\uparrow(\ell) \end{aligned}$$

for $t \in [\tau^\uparrow, \tau^\uparrow + \delta)$. By Proposition 4, we know $\mathcal{X}_t \in \bar{\mathbb{D}}^\uparrow$ for $t \in [\tau^\uparrow, \tau^\uparrow + \delta)$. For $t \notin [\tau^\uparrow, \tau^\uparrow + \delta)$, $\mathcal{X}_t \in \bar{\mathbb{D}}^\uparrow$ following the same argument as that in the proof of Proposition 7. So $Y^{\uparrow''} \in \Pi^\uparrow(\mathcal{X}, Y^{\downarrow'}, \omega)$, a contradiction.

• $A_0 < \frac{w_0^\downarrow(\ell) + w_0^\uparrow(\ell)}{2}$: Similarly, by finding the corresponding δ, ϵ and letting

$$Y^{\downarrow''}(u) = \begin{cases} \begin{cases} Y^{\downarrow'}(\tau^\uparrow) - \epsilon & \text{if } Y^{\downarrow'}(\tau^\uparrow) > Y^\downarrow(\tau^\uparrow), \\ Y^{\downarrow'}(\tau^\uparrow) & \text{if } Y^{\downarrow'}(\tau^\uparrow) = Y^\downarrow(\tau^\uparrow), \end{cases} & u \in [\tau^\uparrow, \tau^\uparrow + \delta), \\ Y^{\downarrow'}(u) & \text{otherwise,} \end{cases}$$

we can show $Y^{\downarrow''} \in \Pi^\downarrow(\mathcal{X}, Y^{\uparrow'}, \omega)$, a contradiction. \square

6.2. The Optimality of the Two-Sided Reflection Policy

In this section, we show that the two-sided reflection mapping $\pi^* = (Y^{\uparrow*}, Y^{\downarrow*})$ is optimal and makes the minimum amount of adjustment to prevent the state $\mathcal{X}_t, t \geq 0$, from falling into \mathbb{D}^\uparrow and \mathbb{D}^\downarrow . Under the one-dimensional setting in Harrison and Taksar [11] and described in Section 5, $\mathbb{D}^\uparrow = \{y < b\}$ and $\mathbb{D}^\downarrow = \{y > a\}$, our two-sided reflection mapping reduces to the same closed forms

$$\begin{aligned} R(t) &= \sup_{0 \leq u \leq t} [a - \omega(u) + L(u)], \quad t \geq 0, \\ L(t) &= \sup_{0 \leq u \leq t} [\omega(u) + R(u) - b], \quad t \geq 0 \end{aligned}$$

in their paper. This reflection mapping makes the minimum amount of adjustment to keep the controlled process in the region $\{a \leq y \leq b\}$.

Theorem 5. *The policy $\pi^* = (Y^{\uparrow*}, Y^{\downarrow*})$ is optimal—that is, $C(\mathcal{X}, \pi^*) = C^*(\mathcal{X})$ for all $\mathcal{X} \in \mathbb{D}$.*

We prove Theorem 5 by considering a cost characterized by $\delta > 0$ in (39) and showing that this cost approaches both $C(\mathcal{X}, \pi^*)$ (Lemma 5) and $C^*(\mathcal{X})$ (Lemma 6) as $\delta \rightarrow 0$. Let \mathcal{X}_t be the state at t under the two-sided reflection policy $\pi^* = (Y^{\uparrow*}, Y^{\downarrow*})$ with initial profile \mathcal{X} . For any small $\delta > 0$, let $\mathbb{D}^\downarrow - \delta = \{\mathcal{X}' - \delta, : \forall \mathcal{X}' \in \mathbb{D}^\downarrow\}$ and $\mathbb{D}^\uparrow + \delta := \{\mathcal{X}' + \delta : \forall \mathcal{X}' \in \mathbb{D}^\uparrow\}$. For a given sample path of the Brownian motion and associated control π^* , the state \mathcal{X}_t will enter

$\mathbb{D}^\uparrow + \delta$ when $\frac{\partial C(\mathcal{X}_t - \delta, 0, 0)}{\partial \xi^\uparrow} = 0$ and $\mathbb{D}^\downarrow - \delta$ when $\frac{\partial C(\mathcal{X}_t + \delta, 0, 0)}{\partial \xi^\downarrow} = 0$ many times over time. Without loss of generality, we assume that \mathcal{X}_t first enters $\mathbb{D}^\uparrow + \delta$ and at

$$\tau_1^\delta = \inf \left\{ t \geq 0 : \frac{\partial C(\mathcal{X}_t - \delta, 0, 0)}{\partial \xi^\uparrow} = 0 \right\}.$$

The process evolves and eventually enters $\mathbb{D}^\downarrow - \delta$ at

$$\tau_2^\delta = \inf \left\{ t > \tau_1^\delta : \frac{\partial C(\mathcal{X}_t + \delta, 0, 0)}{\partial \xi^\downarrow} = 0 \right\}.$$

For $j = 1, 2, \dots$, define

$$\begin{aligned} \tau_{2j+1}^\delta &= \inf \left\{ t > \tau_{2j}^\delta : \frac{\partial C(\mathcal{X}_t - \delta, 0, 0)}{\partial \xi^\uparrow} = 0 \right\}, \\ \tau_{2j+2}^\delta &= \inf \left\{ t > \tau_{2j+1}^\delta : \frac{\partial C(\mathcal{X}_t + \delta, 0, 0)}{\partial \xi^\downarrow} = 0 \right\}; \end{aligned}$$

τ_{2j+1}^δ represents the first time \mathcal{X}_t enters $\mathbb{D}^\uparrow + \delta$ since τ_{2j}^δ , and τ_{2j+2}^δ represents the first time \mathcal{X}_t enters $\mathbb{D}^\downarrow - \delta$ since τ_{2j+1}^δ . Thus, $\tau_i^\delta, i = 1, 2, \dots$, form a series of stopping times. Let $N(t) = \max\{k : \tau_k^\delta \leq t\}$ be the total number of such stopping times by t ,

$$\mathcal{X}_t^\delta = \begin{cases} \mathcal{X}_t - \delta, & \text{if } t < \tau_1^\delta, \\ \mathcal{X}_t + \delta, & \text{if } \tau_{2j-1}^\delta \leq t < \tau_{2j}^\delta, \\ \mathcal{X}_t - \delta, & \text{if } \tau_{2j}^\delta \leq t < \tau_{2j+1}^\delta, \end{cases}$$

and

$$C^\delta(\mathcal{X}, \pi^*) = \mathbb{E} \left[\int_0^\infty e^{-\gamma t} h(\mathcal{X}_t^\delta(0)) dt + k^\uparrow \int_0^\infty e^{-\gamma t} dY^{\uparrow*}(t) + k^\downarrow \int_0^\infty e^{-\gamma t} dY^{\downarrow*}(t) \right] \quad (39)$$

is the cost associated with the process $\{\mathcal{X}_t^\delta\}$ and policy π^* . $C^\delta(\mathcal{X}, \pi^*)$ differs from $C(\mathcal{X}, \pi^*)$ only by the holding cost term, and the difference is bounded by $\int_0^\infty e^{-\gamma t} t dt = \frac{M}{\gamma} \delta$ as stated in the following lemma.

Lemma 5. $|C(\mathcal{X}, \pi^*) - C^\delta(\mathcal{X}, \pi^*)| \leq \frac{M}{\gamma} \delta$.

Applying Proposition 9 and Theorem 3, we can show the following lemma.

Lemma 6. For any fixed $T \geq 0$,

$$C^\delta(\mathcal{X}, \pi^*) \leq C^*(\mathcal{X}) + (2\mathbb{E}N(T) + 3)M\delta - R_1(\mathcal{X}, \delta, T) + R_2(T), \quad (40)$$

where $R_1(\mathcal{X}, \delta, T) \rightarrow 0$ as $\delta \rightarrow 0$ for any fixed T and $R_2(T) \rightarrow 0$ as $T \rightarrow \infty$.

The proof is quite technical and can be found in the appendix. We are now ready to prove the optimality of the two-sided reflection policy π^* .

Proof of Theorem 5. We first show that $\mathbb{E}N(t)$ is finite for any $t \geq 0$. Consider a sequence of stopping times of the Brownian motion W_t ,

$$\begin{aligned} U_1 &= \inf \left\{ t > 0, |W_t| = \frac{w_0^\downarrow(\ell) - w_0^\uparrow(\ell)}{4} \right\}, \\ U_j &= \inf \left\{ t > U_{j-1}, |W_t - W_{U_{j-1}}| = \frac{w_0^\downarrow(\ell) - w_0^\uparrow(\ell)}{4} \right\}, \quad j = 1, 2, \dots \end{aligned}$$

and let $N'(t) = \max\{j : U_j \leq t\}$ be the corresponding counting process.

By the definitions of two consecutive stopping times τ_{2j-1}^δ and τ_{2j}^δ , \mathcal{X}_t^δ enters $\mathbb{D}^\downarrow - \delta$ at τ_{2j-1}^δ and then enters $\mathbb{D}^\uparrow + \delta$ at τ_{2j}^δ . By the same argument leading to (38) in the proof of Proposition 10, for a small enough δ , there exist $\tau_{2j-1}^\delta \leq s_1 < s_2 \leq \tau_{2j}^\delta$ such that

$$W_{s_1} - W_{s_2} \geq w_0^\downarrow(\ell) - w_0^\uparrow(\ell) - 2\delta > \frac{w_0^\downarrow(\ell) - w_0^\uparrow(\ell)}{2}.$$

Thus, there must exist i_j such that $U_{i_j} \in [s_1, s_2] \subset [\tau_{2j-1}^\delta, \tau_{2j}^\delta]$ for each $j = 1, 2, \dots$. Hence, $N(t) \leq 2N'(t)$ for any $t > 0$, and $\mathbb{E}N(t)$ is finite. Fixing the T and letting $\delta \rightarrow 0$ in Lemma 5 and Lemma 6, we have

$$C(\mathcal{X}, \pi^*) \leq C^*(\mathcal{X}) + R_2(T) \tag{41}$$

for any $T \geq 0$. Note that $R_2(T) \rightarrow 0$ as $T \rightarrow \infty$, combining the above with the optimality of $C^*(\mathcal{X})$, we have $C(\mathcal{X}, \pi^*) = C^*(\mathcal{X})$. \square

7. Conclusion and Discussion

In this paper, we consider the optimal control of a storage system whose content is driven by a Brownian motion in the absence of control. Because there is a positive lead time for upward adjustments, the state of the system is a function on a continuous interval, and such a problem is extremely challenging. We develop a novel four-step approach described in the introduction to identify the structure of optimal control as a state-dependent two-sided reflection mapping that makes the minimum amount of upward or downward adjustment to prevent the state from entering into certain regions. To the best of our knowledge, this is the first paper to study instantaneous control of stochastic systems in a functional setting, and the methodology developed in the paper may inspire ways to solve other control problems in various applications.

7.1. The General Case with a Lead Time for Downward Adjustments

We have assumed that downward adjustments are instantaneous. If they are not and there is a positive lead time for downward adjustments, then by the time a promised downward adjustment is made, there may not be enough content left because of the Brownian motion. The only way to avoid this situation completely is to add a constraint on downward adjustments and set aside enough inventory. But then it will be too difficult to calculate the inventory cost.

Now suppose that backlogging of downward adjustments after the lead time is allowed at the same penalty cost as that whenever the content is negative. Then, if the lead times for upward and downward adjustments are identical, the problem can be reduced to one with zero upward and downward adjustment lead times by theorem 3.11 in Øksendal and Sulem [14]. Otherwise, our analysis can be extended by transforming the problem into one with a single lead time as follows. Since upward and downward adjustments are symmetric analytically when the latter can be backlogged, we only need to consider the case where $\ell^\uparrow \geq \ell^\downarrow > 0$ and show that the system can be transformed into one with zero lead time for downward adjustments.

Define $\mathcal{X}_t^i(u)$ as the total outstanding movement i , $i \in \{\uparrow, \downarrow\}$, at time t but before any adjustment at time t that will be realized during $(t, t + u]$ and $\mathcal{X}_t^i = \{\mathcal{X}_t^i(u), u \geq 0\}$. Then, $(\mathcal{X}_t^\uparrow, \mathcal{X}_t^\downarrow)$ is the profile of the outstanding movements at time t with $\mathcal{X}_t^i(0) = 0$ and $\mathcal{X}_t^i(u) = \mathcal{X}_t^i(\ell^i)$ for $u > \ell^i$, and $(H_t, \mathcal{X}_t^\uparrow, \mathcal{X}_t^\downarrow)$ describes the state of the system at time t . Hence, for $t > 0$, the dynamics of the system can be written as

$$H_t = H_0 + W_t + \mathcal{X}_0^\uparrow(t) - \mathcal{X}_0^\downarrow(t) + Y^\uparrow(t - \ell^\uparrow) - Y^\downarrow(t - \ell^\downarrow), \tag{42}$$

$$\mathcal{X}_t^i(u) = \begin{cases} \mathcal{X}_0^i(t + u) - \mathcal{X}_0^i(t) + Y^i(t + u - \ell^i) - Y^i(t - \ell^i), & \text{if } u \leq \ell^i, \\ \mathcal{X}_t^i(\ell^i), & \text{else,} \end{cases} \tag{43}$$

and the cost function for any initial state $(H_0, \mathcal{X}_0^\uparrow, \mathcal{X}_0^\downarrow)$ and policy π is

$$\tilde{C}(H_0, \mathcal{X}_0^\uparrow, \mathcal{X}_0^\downarrow, \pi) = \mathbb{E} \left[\int_0^\infty e^{-\gamma t} h(H_t) dt + \int_0^\infty e^{-\gamma t} k^\uparrow dY^\uparrow(t) + \int_0^\infty e^{-\gamma t} k^\downarrow dY^\downarrow(t) \right]. \tag{44}$$

Now consider another system where there is no lead time for downward adjustments and the lead time for upward adjustments is $\ell = \ell^\uparrow - \ell^\downarrow$, the initial state is $\mathcal{X}_0(u) = H_0 + \mathcal{X}_0^\uparrow(u + \ell^\downarrow) - \mathcal{X}_0^\downarrow(\ell^\downarrow)$, and the holding cost rate is $\tilde{h}(x) = e^{-\gamma \ell^\downarrow} E[h(x + \mathcal{N}_{\ell^\downarrow})]$.

Proposition 11. *For any fixed policy π ,*

$$C(\mathcal{X}_0, \pi) = \tilde{C}(H_0, \mathcal{X}_0^\uparrow, \mathcal{X}_0^\downarrow, \pi) - \mathbb{E} \left[\int_0^{\ell^\downarrow} e^{-\gamma t} h(H_t) dt \right],$$

where $\mathcal{X}_0(u) = H_0 + \mathcal{X}_0^\uparrow(u + \ell^\downarrow) - \mathcal{X}_0^\downarrow(\ell^\downarrow)$ and $\mathbb{E} \left[\int_0^{\ell^\downarrow} e^{-\gamma t} h(H_t) dt \right]$ is a constant for given $(H_0, \mathcal{X}_0^\uparrow, \mathcal{X}_0^\downarrow)$.

This proposition reveals that the difference between the cost functions of the single lead time system and the original system is a constant under the same policy. Thus, the problem reduces to one with zero lead time for downward adjustments.

7.2. Lead Times with General Distribution

In practice, lead times can be random, time dependent, cost dependent, and so on, and indeed inventory systems with various lead time profiles have been studied extensively in the past decades. Those studies built on earlier research on systems with zero lead time and a fixed lead time, and Porteus [17] (also, many papers include those by Nahmias mentioned in the literature review) provides a comprehensive review. In this paper, we characterize the optimal structure for a continuous review system with a fixed lead time where the state space is a functional one, which is extremely challenging and a first step toward the study of more realistic problems. Thus, our contribution is more methodological than a direct application. Below, we discuss the challenges of solving the problem when the lead time is random and cost dependent. When demand is time dependent, the problem is even more difficult.

1. The lead time is random and follows a certain distribution on $[0, \ell]$. In this case, our current functional space will not be enough to describe the state because of the uncertainty of the arrival time of each outstanding order. Furthermore, orders may cross in time—that is, orders placed earlier may arrive later. So one has to first find a way to describe the state and the system dynamics when the lead time is random, which are nontrivial. Even if the lead time possesses the memoryless property, the dimension of the state space remains the same as one still needs to keep track of both the timing and magnitude of each control, which can vary over time.

2. The lead time is cost dependent. Suppose that one has an option to pay a premium for a faster delivery service. We can show that the optimal cost function for any given lead time, long or short, is L^{\natural} -convex and the smaller of two convex functions. However, since the infimum of convex functions is not necessary convex, our structural results on the optimal policy may not hold.

Appendix

Proofs of Lemmas and Propositions

Proof of the completeness of space \mathbb{D} . First, since any function stays constant over $t \geq \ell$, \mathbb{D} is a subset of the complete L^1 space. So, for any Cauchy sequence $\{f_i: i = 1, 2, \dots\}$ in \mathbb{D} , there exists a $f \in L^1$ such that $f(t) = \lim_{i \rightarrow \infty} f_i(t)$ almost surely. Moreover, $f_i(\ell)$ must converge as $i \rightarrow \infty$. Since $f_i \in \mathbb{D}$ is a nondecreasing function for all $i = 1, 2, \dots$ and well defined at each t , $f_{\sup} = \limsup_{i \rightarrow \infty} f_i(t)$ is also nondecreasing, constant for $t \geq \ell$, and continuous almost everywhere. Thus, there exists another nondecreasing function f^* such that $f^*(t) = f_{\sup}(t)$ almost surely and is right-continuous at each t . Since $f(t) = \lim_{i \rightarrow \infty} f_i(t)$ almost surely, we have $f^*(t) = f(t)$ almost surely and hence $f^*(t) \in \mathbb{D}$. \square

Proof of Proposition 1. By the definition of $C^*(\mathcal{X})$, for any $\epsilon > 0$, we can find a policy π such that $C(\mathcal{X}, \pi) \leq C^*(\mathcal{X}) + \epsilon$. We apply the same policy π to the state \mathcal{X}' and denote \mathcal{X}_t and \mathcal{X}'_t to be the states under π with initial state \mathcal{X} and \mathcal{X}' , respectively.

$$\begin{aligned} C^*(\mathcal{X}') - C^*(\mathcal{X}) - \epsilon &\leq C(\mathcal{X}', \pi) - C(\mathcal{X}, \pi) = \mathbb{E} \left[\int_0^{\infty} e^{-\gamma t} [h(\mathcal{X}'_t(0)) - h(\mathcal{X}_t(0))] dt \right] \\ &\leq M \int_0^{\infty} e^{-\gamma t} [\mathcal{X}'(t) - \mathcal{X}(t)] dt = Md(\mathcal{X}', \mathcal{X}). \end{aligned}$$

By symmetry, we also have $C^*(\mathcal{X}) - C^*(\mathcal{X}') - \epsilon \leq Md(\mathcal{X}', \mathcal{X})$. Letting $\epsilon \rightarrow 0$, we have that $C^*(\mathcal{X})$ is Lipschitz continuous. \square

Proof of Proposition 2. For any given state $\mathbf{x} = (x_0, x_1, \dots, x_{n-1})$, if we let $x_n = x_{n-1} + y^{\uparrow}$, we can rewrite

$$\begin{aligned} c_i^{T,n}(\mathbf{x}, y^{\uparrow}, y^{\downarrow}) &= k^{\downarrow} x_n + k^{\uparrow} y^{\downarrow} - k^{\downarrow} x_{n-1} \\ &\quad + \alpha \mathbb{E} [C_{i+1}^{T,n}((x_1, x_2, \dots, x_{n-1}, x_n) - y^{\downarrow} \mathbf{e} + w_t \mathbf{e}) + h^n(x_0 - y^{\downarrow})] \end{aligned}$$

and view $c_i^{T,n}(\mathbf{x}, y^{\uparrow}, y^{\downarrow})$ as a function of $(\mathbf{x}, x_n, y^{\downarrow})$. We next show by induction that $c_i^{T,n}(\mathbf{x}, y^{\uparrow}, y^{\downarrow})$ is L^{\natural} -convex in $(\mathbf{x}, x_n, y^{\downarrow})$ and $C_{i+1}^{T,n}(\mathbf{x})$ is L^{\natural} -convex in \mathbf{x} simultaneously.

Since $h^n(x)$ is convex, $C_T^{T,n}(\mathbf{x})$ is 0 and hence L^{\natural} -convex in \mathbf{x} . Assuming that $C_{i+1}^{T,n}(\mathbf{x})$ is L^{\natural} -convex in \mathbf{x} . Since $h^n(\cdot)$ is convex and $C_{i+1}^{T,n}((x_1, x_2, \dots, x_{n-1}, x_n) - y^{\downarrow} \mathbf{e} + w_t \mathbf{e})$ is L^{\natural} -convex in $(x_1, \dots, x_n, y^{\downarrow})$ for a given w_t , by lemma 1 in Zipkin [24], $c_i^{T,n}(\mathbf{x}, y^{\uparrow}, y^{\downarrow})$ is L^{\natural} -convex in $(\mathbf{x}, x_n, y^{\downarrow})$ as L^{\natural} -convexity is preserved by expectation. Thus,

$$C_i^{T,n}(\mathbf{x}) = \min_{x_n \geq x_{n-1}, y^{\downarrow} \geq 0} \{c_i^{T,n}(\mathbf{x}, y^{\uparrow}, y^{\downarrow})\} = \min_{x_n \geq x_{n-1}} \left\{ \min_{y^{\downarrow} \geq 0} \{c_i^{T,n}(\mathbf{x}, y^{\uparrow}, y^{\downarrow})\} \right\}$$

is L^{\natural} -convex in \mathbf{x} by lemma 2 in Zipkin [24] as minimization over a sublattice preserves L^{\natural} -convexity.

Proof of Lemma 1.

1. Since $\phi(\xi^\uparrow, \xi^\downarrow)$ is a linear function of $(\xi^\uparrow, \xi^\downarrow)$, we only need to show the monotonicity of $e^{\frac{\partial C(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}))}{\partial \xi^\uparrow}}$. For any $\epsilon > 0$ and $\mathcal{X}_1 \leq \mathcal{X}_2$ where $\mathcal{X}_1(\ell) = \mathcal{X}_2(\ell)$, $\Phi_{\xi^\uparrow + \epsilon, \xi^\downarrow}(\mathcal{X}_1) \vee \Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}_2) = \Phi_{\xi^\uparrow + \epsilon, \xi^\downarrow}(\mathcal{X}_2)$ and $\Phi_{\xi^\uparrow + \epsilon, \xi^\downarrow}(\mathcal{X}_1) \wedge \Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}_2) = \Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}_1)$. Since $C^*(\mathcal{X})$ is L^h -convex, letting $\xi_1 = \xi_2 = 0$ and $F = C^*$ in Definition 3, we have

$$C^*(\Phi_{\xi^\uparrow + \epsilon, \xi^\downarrow}(\mathcal{X}_1)) + C^*(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}_2)) \geq C^*(\Phi_{\xi^\uparrow + \epsilon, \xi^\downarrow}(\mathcal{X}_2)) + C^*(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}_1)),$$

or

$$C^*(\Phi_{\xi^\uparrow + \epsilon, \xi^\downarrow}(\mathcal{X}_1)) - C^*(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}_1)) \geq C^*(\Phi_{\xi^\uparrow + \epsilon, \xi^\downarrow}(\mathcal{X}_2)) - C^*(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}_2)),$$

which implies the monotonicity of $\frac{\partial C(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}))}{\partial \xi^\uparrow}$.

2. For any $\epsilon, a > 0$, letting $F = C^*$, $\mathcal{X}_1 = \Phi_{\xi^\uparrow, 0}(\mathcal{X})$, $\mathcal{X}_2 = \Phi_{\xi^\uparrow + \epsilon, 0}(\mathcal{X})$ and $(\xi_1, \xi_2) = (0, -a)$ in Definition 3, we have

$$C^*(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}) - 0) + C^*(\Phi_{\xi^\uparrow + \epsilon, \xi^\downarrow}(\mathcal{X}) - (-a)) \geq C^*(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}) - (-a)) + C^*(\Phi_{\xi^\uparrow + \epsilon, \xi^\downarrow}(\mathcal{X}) - 0),$$

which implies $\frac{\partial C(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}) + a)}{\partial \xi^\uparrow} \geq \frac{\partial C(\Phi_{\xi^\uparrow, \xi^\downarrow}(\mathcal{X}))}{\partial \xi^\uparrow}$, and the result holds.

3. For any $\epsilon > 0$ and $\mathcal{X}_1 \geq \mathcal{X}_2$, letting $F = C^*$ and $(\xi_1, \xi_2) = (\xi, \xi + \epsilon)$ in Definition 3, we have

$$C^*(\mathcal{X}_1 - \xi) + C^*(\mathcal{X}_2 - (\xi + \epsilon)) \geq C^*(\mathcal{X}_2 - \xi) + C^*(\mathcal{X}_1 - (\xi + \epsilon)),$$

which implies $\frac{\partial C(\Phi_{0, \xi}(\mathcal{X}_1))}{\partial \xi^\uparrow} \geq \frac{\partial C(\Phi_{0, \xi}(\mathcal{X}_2))}{\partial \xi^\uparrow}$. Replacing \mathcal{X}_1 and \mathcal{X}_2 by $\Phi_{\xi^\uparrow, 0}(\mathcal{X}_1)$ and $\Phi_{\xi^\uparrow, 0}(\mathcal{X}_2)$, we have that $\frac{\partial C(\mathcal{X}, \xi^\uparrow, \xi^\downarrow)}{\partial \xi^\uparrow}$ is decreasing in \mathcal{X} . \square

Proof of Equations (27) and (28). Note that, under the periodic policy π^n , adjustments can only be made at $T_i^n = \frac{it}{n}$ and at the amounts $(\xi_i^{n\uparrow}, \xi_i^{n\downarrow})$ for $i = 0, 1, 2, \dots$. For convenience, we use T_i to represent T_i^n and $(\xi_i^\uparrow, \xi_i^\downarrow)$ to represent $(\xi_i^{n\uparrow}, \xi_i^{n\downarrow})$ for $i = 0, 1, 2, \dots$ in this proof.

- On the event A , rewrite $C(\mathcal{X}', \pi^n)$ as

$$\mathbb{E} \left[\int_0^{T_{N(\epsilon)}} e^{-\gamma t} h(\mathcal{X}'_t(0)) dt + \sum_{i=1}^{N(\epsilon)} e^{-\gamma T_i} \phi(\xi_i^\uparrow, \xi_i^\downarrow) \right] + \mathbb{E} \left[\int_{T_{N(\epsilon)}}^\infty e^{-\gamma t} h(\mathcal{X}'_t(0)) dt + \sum_{N(\epsilon)+1}^\infty e^{-\gamma T_i} \phi(\xi_i^\uparrow, \xi_i^\downarrow) \right],$$

where the second item is the discounted control cost given initial state $\mathcal{X}'_{T_{N(\epsilon)}}$ and is thus always larger than or equal to the lower bound $\mathbb{E} \left[e^{-\gamma T_{N(\epsilon)}} C^*(\mathcal{X}'_{T_{N(\epsilon)}}) \right]$. Hence, we have

$$C(\mathcal{X}', \pi^n) \geq \mathbb{E} \left[\int_0^{T_{N(\epsilon)}} e^{-\gamma t} h(\mathcal{X}'_t(0)) dt + \sum_{i=1}^{N(\epsilon)} e^{-\gamma T_i} \phi(\xi_i^\uparrow, \xi_i^\downarrow) \right] + \mathbb{E} \left[e^{-\gamma T_{N(\epsilon)}} C^*(\mathcal{X}'_{T_{N(\epsilon)}}) \right]. \tag{A.1}$$

Note that the optimal cost $C^*(\mathcal{X}')$ can be written as

$$C^*(\mathcal{X}') = \mathbb{E} \left[\sum_{i=1}^{N(\epsilon)} [e^{-\gamma T_{i-1}} C^*(\mathcal{X}'_{T_{i-1}}) - e^{-\gamma T_i} C^*(\mathcal{X}'_{T_i})] \right] + \mathbb{E} \left[\sum_{i=0}^{N(\epsilon)} e^{-\gamma T_i} [C^*(\mathcal{X}'_{T_i}) - C^*(\mathcal{X}'_{T_i})] \right] + \mathbb{E} \left[e^{-\gamma T_{N(\epsilon)}} C^*(\mathcal{X}'_{T_{N(\epsilon)}}) \right]. \tag{A.2}$$

By the optimality condition (6), the first term in (A.2) is smaller than

$$\sum_{i=1}^{N(\epsilon)} \mathbb{E} \left[\int_{T_{i-1}}^{T_i} e^{-\gamma t} h(\mathcal{X}'_t(0) + W_t) dt \right] = \mathbb{E} \left[\int_0^{T_{N(\epsilon)}} e^{-\gamma t} h(\mathcal{X}'_t(0) + W_t) dt \right]. \tag{A.3}$$

By the dynamics (2) and the definition of $C(\mathcal{X}, \xi^\uparrow, \xi^\downarrow)$, we have $C^*(\mathcal{X}'_{T_i}) = C^*(\Phi_{\xi_i^\uparrow, \xi_i^\downarrow}(\mathcal{X}'_{T_{i-1}})) = C(\mathcal{X}'_{T_{i-1}}, \xi_i^\uparrow, \xi_i^\downarrow) - \phi(\xi_i^\uparrow, \xi_i^\downarrow)$, and the second term in (A.2) can be written as

$$\mathbb{E} \left[\sum_{i=0}^{N(\epsilon)} e^{-\gamma T_i} [C(\mathcal{X}'_{T_i}, 0, 0) - C(\mathcal{X}'_{T_i}, \xi_i^\uparrow, \xi_i^\downarrow)] \right] + \mathbb{E} \left[\sum_{i=0}^{N(\epsilon)} e^{-\gamma T_i} \phi(\xi_i^\uparrow, \xi_i^\downarrow) \right]. \tag{A.4}$$

Let A denote the event where $\{T_{N(\epsilon)} \leq \tau\}$ and A^c its complement, and $\mathbb{E}_A[X] = \mathbb{E}[X1_A]$ for any random variable X . Then (A.4) can be bounded from above by

$$\mathbb{E}_A \left[\sum_{i=0}^{N(\epsilon)} e^{-\gamma T_i} [C(\mathcal{X}'_{T_i}, 0, 0) - C(\mathcal{X}'_{T_i}, \xi_i^\uparrow, \xi_i^\downarrow)] \right] + \mathbb{E} \left[\sum_{i=0}^{N(\epsilon)} e^{-\gamma T_i} \phi(\xi_i^\uparrow, \xi_i^\downarrow) \right], \tag{A.5}$$

after dropping the term $\mathbb{E}_{A^c}[\cdot]$. Since $C(\mathcal{X}, 0, 0) - C(\mathcal{X}, \xi^\uparrow, \xi^\downarrow)$ is always nonpositive for any $(\mathcal{X}, \xi^\uparrow, \xi^\downarrow)$ by Proposition 4, and $C(\mathcal{X}, \xi^\uparrow, \xi^\downarrow)$ is convex in ξ^\uparrow and ξ^\downarrow , respectively,

$$\begin{aligned} & C(\mathcal{X}'_{T_i-}, 0, 0) - C(\mathcal{X}'_{T_i-}, \xi_i^\uparrow, \xi_i^\downarrow) \\ &= [C(\mathcal{X}'_{T_i-}, 0, 0) - C(\mathcal{X}'_{T_i-}, \xi_i^\uparrow, 0)] + [C(\mathcal{X}'_{T_i-}, \xi_i^\uparrow, 0) - C(\mathcal{X}'_{T_i-}, \xi_i^\uparrow, \xi_i^\downarrow)] \\ &\leq -\frac{\partial C(\mathcal{X}'_{T_i-}, \xi_i^\uparrow, 0)}{\partial \xi_i^\uparrow} \xi_i^\uparrow - \frac{\partial C(\mathcal{X}'_{T_i-}, \xi_i^\uparrow, \xi_i^\downarrow)}{\partial \xi_i^\downarrow} \xi_i^\downarrow \\ &= -\frac{\partial C(\Phi_{\xi_i^\uparrow, 0}(\mathcal{X}'_{T_i-}), 0, 0)}{\partial \xi_i^\uparrow} \xi_i^\uparrow - \frac{\partial C(\mathcal{X}'_{T_i-}, 0, 0)}{\partial \xi_i^\downarrow} \xi_i^\downarrow. \end{aligned} \tag{A.6}$$

On the event A , for any $k \leq N(\epsilon)$, (W_{T_i}, T_i) is in the set $(\hat{w} - w, \hat{s} - s) + B(\delta)$. Consequently, $(w + W_{T_i}, s + T_i)$ is in $(\hat{w}, \hat{s}) + B(\delta)$. Moreover, the cumulative amount of upward and downward adjustments at time T_i is less than δ , which means $\sum_{i \leq k} \xi_i^\uparrow + \sum_{i \leq k} \xi_i^\downarrow \leq \epsilon \leq \delta$. By (2),

$$\begin{aligned} d(\sigma_{\hat{s}}(\mathcal{X}) + \hat{w}, \mathcal{X}'_{T_i}) &\leq d(\sigma_{\hat{s}}(\mathcal{X}) + \hat{w}, \sigma_{(s+T_i)}(\mathcal{X}) + w + W_i) + \sum_{i \leq k} \xi_i^\uparrow + \sum_{i \leq k} \xi_i^\downarrow \\ &\leq d(\sigma_{\hat{s}}(\mathcal{X}) + \hat{w}, \sigma_{(s+T_i)}(\mathcal{X}) + \hat{w}) + d(\sigma_{(s+T_i)}(\mathcal{X}) + \hat{w}, \sigma_{(s+T_i)}(\mathcal{X}) + w + W_i) + \delta \\ &\leq (s + T_i - \hat{s})\gamma\mathcal{X}(\ell) + 2\delta \leq 3\delta. \end{aligned}$$

Similarly, we have $d(\sigma_{\hat{s}}(\mathcal{X}) + \hat{w}, \Phi_{\xi_i^\uparrow, 0}(\mathcal{X}'_{T_i-})) \leq 3\delta$. Thus, by (24), we have $\frac{\partial C(\Phi_{\xi_i^\uparrow, 0}(\mathcal{X}'_{T_i-}), 0, 0)}{\partial \xi_i^\uparrow} \geq k_0$ and $\frac{\partial C(\mathcal{X}'_{T_i-}, 0, 0)}{\partial \xi_i^\downarrow} \geq k_0$. That is, (A.6) is bounded by $-k_0(\xi_i^\uparrow + \xi_i^\downarrow)$ on the event A . Consequently, the first term in (A.5) is bounded from above by

$$\mathbb{E}_A \left[\sum_{i=0}^{N(\epsilon)} -e^{-\gamma\delta} k_0 (\xi_i^\uparrow + \xi_i^\downarrow) \right] \leq -\mathbb{P}(A) e^{-\gamma\delta} k_0 \epsilon. \tag{A.7}$$

Plugging (A.3), (A.5), and (A.7) into (A.2), we have

$$C^*(\mathcal{X}') \leq \mathbb{E} \left[\int_0^{T_{N(\epsilon)}} e^{-\gamma t} h(\mathcal{X}'_t(0)) dt + \sum_{i=1}^{N(\epsilon)} e^{-\gamma T_i} \phi(\xi_i^\uparrow, \xi_i^\downarrow) + e^{-\gamma T_{N(\epsilon)}} C^*(\mathcal{X}'_{T_{N(\epsilon)}}) \right] - \mathbb{P}(A) e^{-\gamma\delta} k_0 \epsilon.$$

Comparing it with (A.1), we have

$$C(\mathcal{X}', \pi^n) \geq C^*(\mathcal{X}') + \mathbb{P}(A) e^{-\gamma\delta} k_0 \epsilon \geq V_{\mathcal{X}'}(0, 0) + \mathbb{P}(A) e^{-\gamma\delta} k_0 \epsilon.$$

- On the event A^c , rewrite $C(\mathcal{X}', \pi^n)$ as

$$\mathbb{E} \left[\int_0^{\tau'} e^{-\gamma t} h(\mathcal{X}'_t(0)) dt + \sum_{T_i \leq \tau'} e^{-\gamma T_i} \phi(\xi_i^\uparrow, \xi_i^\downarrow) \right] + \mathbb{E} \left[\int_{\tau'}^\infty e^{-\gamma t} h(\mathcal{X}'_t(0)) dt + \sum_{T_i > \tau'} e^{-\gamma T_i} \phi(\xi_i^\uparrow, \xi_i^\downarrow) \right].$$

Similar to the argument in (A.1), the second term is greater than $\mathbb{E} \left[e^{-\gamma \tau'} C^*(\mathcal{X}'_{\tau'}) \right]$. Dropping the nonnegative item $\mathbb{E}_A[\cdot]$ in the expectations, we have

$$\begin{aligned} C(\mathcal{X}', \pi) &\geq \mathbb{E}_{A^c} \left[\int_0^{\tau'} e^{-\gamma t} h(\mathcal{X}'_t(0)) dt + \sum_{T_i \leq \tau'} e^{-\gamma T_i} \phi(\xi_i^\uparrow, \xi_i^\downarrow) \right] + \mathbb{E}_{A^c} \left[e^{-\gamma \tau'} C^*(\mathcal{X}'_{\tau'}) \right] \\ &\geq \mathbb{E}_{A^c} \left[\int_0^{\tau'} e^{-\gamma t} h(\sigma_t(\mathcal{X}')(0) + W_t) dt \right] + \mathbb{E}_{A^c} \left[e^{-\gamma \tau'} C^*(\sigma_{\tau'}(\mathcal{X}') + W_{\tau'}) \right] - \frac{M\epsilon}{\gamma}. \end{aligned} \tag{A.8}$$

The second inequality follows because, on the event A^c , the cumulative amount of upward and downward adjustments by the stopping time τ' is less than ϵ . Thus, by (2), the distance $d(\sigma_s(\mathcal{X}') + W_s, \mathcal{X}'_s) < \epsilon$ for any $0 \leq s \leq \tau'$. By Assumption 1, $|h(\mathcal{X}'_s(0)) - h(\sigma_s(\mathcal{X}')(0) + W_s)| \leq M\epsilon$, and by Proposition 1, $|C^*(\sigma_s(\mathcal{X}') + W_s) - C^*(\mathcal{X}'_s)| \leq \frac{M}{\gamma} d(\sigma_s(\mathcal{X}') + W_s, \mathcal{X}'_s) < \frac{M}{\gamma} \epsilon$ for any $0 \leq s \leq \tau'$. For each of the expectation $\mathbb{E}_{A^c}[\cdot]$ in (A.8), we can write it as the difference $\mathbb{E}[\cdot] - \mathbb{E}_A[\cdot]$. Since the process (W_t, t) does not go out of $(\hat{w} - w, \hat{s} - s) + B(\delta)$ before the stopping time τ' , the shifted process $(w + W_t, s + t)$ is always in $(\hat{w}, \hat{s}) + B(\delta)$ for all $0 \leq t \leq \tau'$. Then, for the $\mathbb{E}_A[\cdot]$ terms, we have the following bound:

$$\begin{aligned} & \mathbb{E}_A \left[\int_0^{\tau'} e^{-\gamma t} h(\sigma_t(\mathcal{X}')(0) + W_t) dt \right] + \mathbb{E}_A \left[e^{-\gamma \tau'} C^*(\sigma_{\tau'}(\mathcal{X}') + W_{\tau'}) \right] \\ &= \mathbb{E}_A \left[\int_0^{\tau'} e^{-\gamma t} h(\sigma_{s+t}(\mathcal{X})(0) + w + W_t) dt \right] + \mathbb{E}_A \left[e^{-\gamma \tau'} V_{\mathcal{X}}(w + W_{\tau'}, s + \tau') \right] \\ &\leq \mathbb{P}(A) \int_0^\delta \bar{h} dt + \mathbb{P}(A) \bar{V} = \mathbb{P}(A) (\delta \bar{h} + \bar{V}), \end{aligned} \tag{A.9}$$

where $\bar{h} = \sup_{(w,s) \in (\hat{w}, \hat{s}) + B(\delta)} \{h(\sigma_s(\mathcal{X})(0) + w)\} < \infty$ and $\bar{V} = \sup_{(w,s) \in (\hat{w}, \hat{s}) + B(\delta)} \{V_{\mathcal{X}}(w, s)\} < \infty$, all independent of τ' . This means that $C^*(\sigma_{\tau'}(\mathcal{X}') + W_{\tau'}) = V_{\mathcal{X}}(w + W_{\tau'}, s + \tau') \leq \bar{V}$ and $h(\sigma_t(\mathcal{X}')(0) + W_t) = h(\sigma_{s+t}(\mathcal{X}')(0) + w + W_t) \leq \bar{h}$ for all $t \leq \tau'$. Plugging (A.9) into (A.8), we have

$$C(\mathcal{X}', \pi) \geq \mathbb{E} \left[\int_0^{\tau'} e^{-\gamma t} h(\sigma_t(\mathcal{X}')(0) + W_t) dt \right] + \mathbb{E} \left[e^{-\gamma \tau'} C^*(\sigma_{\tau'}(\mathcal{X}') + W_{\tau'}) \right] - \frac{M\epsilon}{\gamma} - \mathbb{P}(A)(\delta \bar{h} + \bar{V}).$$

Comparing the above with (26), we have

$$C(\mathcal{X}', \pi) \geq V_{\mathcal{X}'}(0, 0) + c_0 - \frac{M\epsilon}{\gamma} - \mathbb{P}(A)(\delta \bar{h} + \bar{V}). \quad \square$$

Proof of Proposition 6. Since $C^*(\mathcal{X})$ is L^1 -convex, the partial derivatives $\frac{\partial V_{\mathcal{X}}(w, s)}{\partial w}$ and $\frac{\partial^2 V_{\mathcal{X}}(w, s)}{\partial w^2}$ exist almost everywhere. Moreover, $\frac{\partial V_{\mathcal{X}}(w, s)}{\partial w} = \frac{\partial C^*(\sigma_s(\mathcal{X}) + w)}{\partial \xi^1}$. By part 3 of Lemma 1, $\frac{\partial V_{\mathcal{X}}(w, s)}{\partial w}$ monotone in s . So the partial derivatives $\frac{\partial^2 V_{\mathcal{X}}(w, s)}{\partial s \partial w}$ exists almost everywhere and hence $\frac{\partial V_{\mathcal{X}}(w, s)}{\partial s}$ exists almost everywhere. Then by the optimality condition we have

$$V_{\mathcal{X}}(w, s) \leq \mathbb{E} \left[\int_0^{\tau'} e^{-\gamma t} h(\mathcal{X}(s+t) + w + W_t) dt \right] + \mathbb{E} \left[e^{-\gamma \tau'} V_{\mathcal{X}}(w + W_{\tau'}, s + \tau') \right]$$

for any stopping time τ' . Combining with the existence of above three the partial derivatives, we immediately derive that

$$\frac{\partial V_{\mathcal{X}}(w, s)}{\partial s} + \frac{\sigma^2}{2} \frac{\partial^2 V_{\mathcal{X}}(w, s)}{\partial w^2} + \mu \frac{\partial V_{\mathcal{X}}(w, s)}{\partial w} - \gamma V_{\mathcal{X}}(w, s) + h(\mathcal{X}(s) + w) \geq 0$$

holds for almost every $(w, s) \in \mathbb{R} \times \mathbb{R}_+$. \square

Proof of Proposition 9. We only prove the result for $\psi^\dagger(\mathcal{X}, Y^\dagger, \omega)$. Suppose the above equation does not hold—i.e., there exists t such that $\frac{\partial C(\mathcal{X}_t, 0, 0)}{\partial \xi^1} > 0$ and ψ^\dagger increases at t .

If $\psi^\dagger(t) > \psi^\dagger(t-)$, then there must exist $\epsilon, \delta > 0$ such that $\psi^\dagger(t) - \psi^\dagger(t-) > \epsilon$ and, for any $\mathcal{X}' \in \mathbb{D}$ that satisfies $\rho_\gamma(\mathcal{X}', \mathcal{X}_t) < \epsilon + \delta \frac{\mathcal{X}_t(t)}{\gamma}$ and $\frac{\partial C(\mathcal{X}', 0, 0)}{\partial \xi^1} > 0$. Hence, the following upward adjustment,

$$Y^{\dagger'}(u) = \begin{cases} \psi^\dagger(u) - \epsilon, & u \in [t, t + \delta), \\ \psi^\dagger(u), & \text{otherwise} \end{cases}$$

is strictly less than ψ^\dagger . Following a similar argument as in the proof of Proposition 8, we can show that $Y^{\dagger'} \in \Pi^\dagger(\mathcal{X}, Y^\dagger, \omega)$, which implies that ψ^\dagger cannot be the infimum, a contradiction.

If $\psi^\dagger(t) = \psi^\dagger(t-)$, there must exist $\epsilon, \delta > 0$ such that $\psi^\dagger(t) - \psi^\dagger(s-) > \epsilon$ and $\frac{\partial C(\mathcal{X}_s, 0, 0)}{\partial \xi^1} > 0$ for $t \leq s \leq t + \delta$. Then, the following upward adjustment,

$$Y^{\dagger'}(u) = \begin{cases} \psi^\dagger(t), & u \in [t, t + \delta), \\ \psi^\dagger(u), & \text{otherwise} \end{cases}$$

is strictly less than ψ^\dagger . Similarly, we can show that $Y^{\dagger'} \in \Pi^\dagger(\mathcal{X}, Y^\dagger, \omega)$, which implies that ψ^\dagger cannot be the infimum, again a contradiction. Thus, the proposition holds. \square

Proof of Lemma 6. The proof is quite complicated, thus we give a road map. Essentially, we prove that, for any fixed $T > 0$,

$$\begin{aligned} & C^\delta(\mathcal{X}, \pi^*) - C^*(\mathcal{X}_0^\delta) + \mathbb{E} \left[e^{-\gamma T} C^*(\mathcal{X}_T^\delta) \right] - (2\mathbb{E}N(T) + 2)M\delta \\ & + \mathbb{E} \left[\int_0^T e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^1} dY^{\dagger*}(t) + \int_0^T e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^1} dY^{\dagger*}(t) \right] \\ & \leq \mathbb{E} \left[\int_T^\infty e^{-\gamma t} h(\mathcal{X}_t^\delta(0)) dt + k^\dagger \int_T^\infty e^{-\gamma t} dY^{\dagger*}(t) + k^\dagger \int_T^\infty e^{-\gamma t} dY^{\dagger*}(t) \right]. \end{aligned} \tag{A.10}$$

Once this is proven, let $R_1(\mathcal{X}, \delta, T) = \mathbb{E} \left[\int_0^T e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^1} dY^{\dagger*}(t) + \int_0^T e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^1} dY^{\dagger*}(t) \right]$ and $R_2(T) = \mathbb{E} \left[\int_T^\infty e^{-\gamma t} h(\mathcal{X}_t^\delta(0)) dt + k^\dagger \int_T^\infty e^{-\gamma t} dY^{\dagger*}(t) + k^\dagger \int_T^\infty e^{-\gamma t} dY^{\dagger*}(t) \right] - \mathbb{E} \left[e^{-\gamma T} C^*(\mathcal{X}_T^\delta) \right]$. Then (A.10) becomes

$$C^\delta(\mathcal{X}, \pi^*) \leq C^*(\mathcal{X}_0^\delta) + (2\mathbb{E}N(T) + 2)M\delta - R_1(\mathcal{X}, \delta, T) + R_2(T). \tag{A.11}$$

By (4) and the Lipschitz continuity of $C^*(\mathcal{X})$, we immediately get that $R_2(T) \rightarrow 0$ as $T \rightarrow \infty$. For $R_1(\mathcal{X}, \delta, T)$, it is easy to see that $\mathcal{X}_t^\delta \rightarrow \mathcal{X}_t$ as $\delta \rightarrow 0$, so $\frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^1}$ converges to $\frac{\partial C(\mathcal{X}_t, 0, 0)}{\partial \xi^1}$ by part 1 of Lemma 1. By Lebesgue's dominated convergence theorem, the upward adjustment cost $\mathbb{E} \left[\int_0^T e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^1} dY^{\dagger*}(t) \right]$ converges to $\mathbb{E} \left[\int_0^T e^{-\gamma t} \frac{\partial C(\mathcal{X}_t, 0, 0)}{\partial \xi^1} dY^{\dagger*}(t) \right]$, which equals to 0 by

Proposition 9. Similarly, for the downward adjustment cost, we have that $\mathbb{E} \left[\int_0^T e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^1} dY^{1*}(t) \right]$ converges to 0. Thus, $R_1(\mathcal{X}, \delta, T) \rightarrow 0$ as $\delta \rightarrow 0$. Finally, since $|C^*(\mathcal{X}) - C^*(\mathcal{X}_0^\delta)| \leq M\delta$, the lemma holds.

The remainder of this proof is devoted to showing (A.10). To this end, we apply the following double telescoping to $C^*(\mathcal{X}_T^\delta) - \mathbb{E} [e^{-\gamma T} C^*(\mathcal{X}_T^\delta)]$ to approximate $C^\delta(\mathcal{X}, \pi^*)$.

1. In the first telescoping, we write $C^*(\mathcal{X}_0^\delta) - \mathbb{E} [e^{-\gamma T} C^*(\mathcal{X}_T^\delta)]$ according to the partition of the interval $[0, T]$ by $0 = \tau_0^\delta < \tau_1^\delta < \dots < \tau_{N(T)}^\delta \leq T$.

$$\begin{aligned} & C^*(\mathcal{X}_0^\delta) - \mathbb{E} [e^{-\gamma T} C^*(\mathcal{X}_T^\delta)] \\ &= \mathbb{E} \sum_{k=1}^{N(T)} \left[e^{-\gamma \tau_{k-1}^\delta} C^*(\mathcal{X}_{\tau_{k-1}^\delta}^\delta) - e^{-\gamma \tau_k^\delta} C^*(\mathcal{X}_{\tau_k^\delta}^\delta) \right] + \mathbb{E} \left[e^{-\gamma \tau_{N(T)}^\delta} C^*(\mathcal{X}_{\tau_{N(T)}^\delta}^\delta) - e^{-\gamma T} C^*(\mathcal{X}_T^\delta) \right] \\ &= \mathbb{E} \sum_{k=1}^{N(T)} \left[\mathbb{E} \left[e^{-\gamma \tau_{k-1}^\delta} C^*(\mathcal{X}_{\tau_{k-1}^\delta}^\delta) - e^{-\gamma \tau_k^\delta} C^*(\mathcal{X}_{\tau_k^\delta}^\delta) \middle| \mathcal{F}_{\tau_{k-1}^\delta}^\delta \right] \right] \end{aligned} \tag{A.12}$$

$$+ \mathbb{E} \left[\mathbb{E} \left[e^{-\gamma \tau_{N(T)}^\delta} C^*(\mathcal{X}_{\tau_{N(T)}^\delta}^\delta) - e^{-\gamma T} C^*(\mathcal{X}_T^\delta) \middle| \mathcal{F}_{\tau_{N(T)}^\delta}^\delta \right] \right]. \tag{A.13}$$

2. Next, we examine all of the terms in (A.12) and (A.13), and apply a subtelescoping on each of them. We construct a partition of the interval $[\tau_{k-1}^\delta, \tau_k^\delta]$ by $\tau_{k-1}^\delta = l_{k,0} < l_{k,1} < \dots < l_{k,j_k} = \tau_k^\delta$ for any $0 < \epsilon < \delta$ where

$$\begin{aligned} \hat{l}_{k,j} &= \inf \left\{ u : u > l_{k,j-1}, (Y^{1*}(u) - Y^{1*}(l_{k,j-1})) \vee (Y^{1*}(u) - Y^{1*}(l_{k,j-1})) \geq \frac{\epsilon}{2} \right\}, \\ l_{k,j} &= \hat{l}_{k,j} \wedge (l_{k,j-1} + \epsilon) \wedge \tau_{k+1}^\delta, \end{aligned}$$

for $j = 1, 2, \dots, j_k$. It is obvious that j_k is almost surely finite. We define $Y^{1\epsilon}$ and $Y^{1\epsilon}$ piecewisely on the interval $[\tau_{k-1}^\delta, \tau_k^\delta]$ as $Y^{1\epsilon}(u) = Y^{1*}(l_{k,j})$ and $Y^{1\epsilon}(u) = Y^{1*}(l_{k,j})$ for $l_{k,j} \leq u < l_{k,j+1}$. It is obvious that they are step functions with jump sizes bounded by $\frac{\epsilon}{2}$. Let \mathcal{X}_t^ϵ be the state at time t under policy $(Y^{1\epsilon}, Y^{1\epsilon})$ with the initial profile \mathcal{X} and define

$$\mathcal{X}_t^{\delta, \epsilon} = \begin{cases} \mathcal{X}_t^\epsilon - \delta, & \text{if } t < \tau_1^\delta, \\ \mathcal{X}_t^\epsilon + \delta, & \text{if } \tau_{2j-1}^\delta \leq t < \tau_{2j}^\delta, \\ \mathcal{X}_t^\epsilon - \delta, & \text{if } \tau_{2j}^\delta \leq t < \tau_{2j+1}^\delta. \end{cases}$$

For $k = 1, 2, \dots, N(t)$, based on the second step of telescoping, we estimate (A.12) as

$$\begin{aligned} & \mathbb{E} \left[\mathbb{E} \left[e^{-\gamma \tau_{k-1}^\delta} C^*(\mathcal{X}_{\tau_{k-1}^\delta}^\delta) - e^{-\gamma \tau_k^\delta} C^*(\mathcal{X}_{\tau_k^\delta}^\delta) \middle| \mathcal{F}_{\tau_{k-1}^\delta}^\delta \right] \right] \\ &= \mathbb{E} \left[e^{-\gamma \tau_{k-1}^\delta} \left[C^*(\mathcal{X}_{\tau_{k-1}^\delta}^\delta) - C^*(\mathcal{X}_{\tau_{k-1}^\delta}^{\delta, \epsilon}) \right] \right] - \mathbb{E} \left[e^{-\gamma \tau_k^\delta} \left[C^*(\mathcal{X}_{\tau_k^\delta}^\delta) - C^*(\mathcal{X}_{\tau_k^\delta}^{\delta, \epsilon}) \right] \middle| \mathcal{F}_{\tau_{k-1}^\delta}^\delta \right] \\ &+ \mathbb{E} \left[\mathbb{E} \left[e^{-\gamma \tau_{k-1}^\delta} C^*(\mathcal{X}_{\tau_{k-1}^\delta}^{\delta, \epsilon}) - e^{-\gamma \tau_k^\delta} C^*(\mathcal{X}_{\tau_k^\delta}^{\delta, \epsilon}) \middle| \mathcal{F}_{\tau_{k-1}^\delta}^\delta \right] \right] \geq -2M\epsilon + \mathbb{E} \left[\mathbb{E} \left[e^{-\gamma \tau_{k-1}^\delta} C^*(\mathcal{X}_{\tau_{k-1}^\delta}^{\delta, \epsilon}) - e^{-\gamma \tau_k^\delta} C^*(\mathcal{X}_{\tau_k^\delta}^{\delta, \epsilon}) \middle| \mathcal{F}_{\tau_{k-1}^\delta}^\delta \right] \right] \\ &= -2M\epsilon + \mathbb{E} \left[\mathbb{E} \left[\sum_{j=1}^{j_k} e^{-\gamma l_{k,j-1}} C^*(\mathcal{X}_{l_{k,j-1}}^{\delta, \epsilon}) - e^{-\gamma l_{k,j}} C^*(\mathcal{X}_{l_{k,j}}^{\delta, \epsilon}) \middle| \mathcal{F}_{\tau_{k-1}^\delta}^\delta \right] \right] + \mathbb{E} \left[\sum_{j=1}^{j_k} e^{-\gamma l_{k,j}} C^*(\mathcal{X}_{l_{k,j}}^{\delta, \epsilon}) - e^{-\gamma l_{k,j}} C^*(\mathcal{X}_{l_{k,j}}^{\delta, \epsilon}) \middle| \mathcal{F}_{\tau_{k-1}^\delta}^\delta \right]. \end{aligned} \tag{A.14}$$

The last equality follows as a result of telescoping on the partition $\tau_{k-1}^\delta = l_{k,0} < l_{k,1} < \dots < l_{k,j_k} = \tau_k^\delta$. Since there is no upward or downward adjustment during $[l_{k,j-1}, l_{k,j}]$, the second term in (A.14) becomes

$$\begin{aligned} & \mathbb{E} \left[\mathbb{E} \left[\sum_{j=1}^{j_k} e^{-\gamma l_{k,j-1}} \mathbb{E} \left[C^*(\mathcal{X}_{l_{k,j-1}}^{\delta, \epsilon}) - e^{-\gamma(l_{k,j} - l_{k,j-1})} C^*(\mathcal{X}_{l_{k,j-1}}^{\delta, \epsilon}) \middle| \mathcal{F}_{l_{k,j-1}} \right] \middle| \mathcal{F}_{\tau_{k-1}^\delta}^\delta \right] \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\sum_{j=1}^{j_k} e^{-\gamma l_{k,j-1}} \mathbb{E} \left[V_{\mathcal{X}_{l_{k,j-1}}^{\delta, \epsilon}}^{\delta, \epsilon}(0, 0) - e^{-\gamma(l_{k,j} - l_{k,j-1})} V_{\mathcal{X}_{l_{k,j-1}}^{\delta, \epsilon}}^{\delta, \epsilon}(W_{l_{k,j}} - W_{l_{k,j-1}}, l_{k,j} - l_{k,j-1}) \middle| \mathcal{F}_{l_{k,j-1}} \right] \middle| \mathcal{F}_{\tau_{k-1}^\delta}^\delta \right] \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\sum_{j=1}^{j_k} e^{-\gamma l_{k,j-1}} \mathbb{E} \left[\int_0^{l_{k,j} - l_{k,j-1}} e^{-\gamma u} h(\mathcal{X}_{l_{k,j-1}}^{\delta, \epsilon}(u) + W_u) du \middle| \mathcal{F}_{l_{k,j-1}} \right] \middle| \mathcal{F}_{\tau_{k-1}^\delta}^\delta \right] \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\sum_{j=1}^{j_k} \int_{l_{k,j-1}}^{l_{k,j}} e^{-\gamma u} h(\mathcal{X}_u^{\delta, \epsilon}(0)) du \middle| \mathcal{F}_{\tau_{k-1}^\delta}^\delta \right] \right] \\ &= \mathbb{E} \left[\int_{\tau_{k-1}^\delta}^{\tau_k^\delta} e^{-\gamma u} h(\mathcal{X}_u^{\delta, \epsilon}(0)) du \right] \rightarrow \mathbb{E} \left[\int_{\tau_{k-1}^\delta}^{\tau_k^\delta} e^{-\gamma u} h(\mathcal{X}_u^\delta(0)) du \right] \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

By the definition of \mathcal{X}_t^δ and $\mathcal{X}_t^{\delta,\epsilon}$, we have $\mathcal{X}_t^{\delta,\epsilon} \in \Xi$ for any $t \geq 0$, which allows us to apply Theorem 3 and Corollary 1 to the second equality with $\mathcal{X}' = \mathcal{X}_{t_{k,j-1}}^{\delta,\epsilon}$ as an initial state. Since $\mathcal{X}_u^{\delta,\epsilon}(0) \rightarrow \mathcal{X}_u^\delta(0)$ and $h(\mathcal{X}_u^{\delta,\epsilon}(0))$ is dominated by $h(\mathcal{X}_t^\delta(0)) + M\delta$ as $\epsilon \rightarrow 0$, convergence is established by Lebesgue’s dominated convergence theorem.

Denote $\Delta_{k,j}^\uparrow = Y^{\uparrow\epsilon}(t_{k,j}) - Y^{\uparrow\epsilon}(t_{k,j-})$ and $\Delta_{k,j}^\downarrow = Y^{\downarrow\epsilon}(t_{k,j}) - Y^{\downarrow\epsilon}(t_{k,j-})$. Then, the third term in (A.14) can be written as

$$\begin{aligned} & \mathbb{E} \left[\sum_{j=1}^{j_k} e^{-\gamma t_{k,j}} C \left(\mathcal{X}_{t_{k,j-}}^{\delta,\epsilon}, \Delta_{k,j}^\uparrow, \Delta_{k,j}^\downarrow \right) - e^{-\gamma t_{k,j}} C \left(\mathcal{X}_{t_{k,j-}}^{\delta,\epsilon}, 0, 0 \right) \right] \\ & + \mathbb{E} \left[\sum_{j=1}^{j_k} e^{-\gamma t_{k,j}} \phi \left(\Delta_{k,j}^\uparrow, \Delta_{k,j}^\downarrow \right) \right] + \mathbb{E} \left[e^{-\gamma t_{k,j}} \left[C^* \left(\mathcal{X}_{t_{k,j_k}}^{\delta,\epsilon} \right) - C^* \left(\mathcal{X}_{t_{k,j_k}}^{\delta,\epsilon} + (-1)^k 2\delta \right) \right] \right] \end{aligned} \tag{A.15}$$

$$\begin{aligned} & \geq \mathbb{E} \left[\sum_{j=1}^{j_k} e^{-\gamma t_{k,j}} C \left(\mathcal{X}_{t_{k,j-}}^{\delta,\epsilon}, \Delta_{k,j}^\uparrow, \Delta_{k,j}^\downarrow \right) - e^{-\gamma t_{k,j}} C \left(\mathcal{X}_{t_{k,j-}}^{\delta,\epsilon}, 0, 0 \right) \right] \\ & + \mathbb{E} \left[\sum_{j=1}^{j_k} e^{-\gamma t_{k,j}} \phi \left(\Delta_{k,j}^\uparrow, \Delta_{k,j}^\downarrow \right) \right] - 2M\delta, \end{aligned} \tag{A.16}$$

where the last term in (43) is due to the fact that, in addition to the jumps $(\Delta_{k,j_k}^\uparrow, \Delta_{k,j_k}^\downarrow)$, $\mathcal{X}_t^{\delta,\epsilon}$ also includes the jump caused by δ at $t_{k,j_k} = \tau_k^\delta$. The second term in (A.16) is the total discounted ordering cost under policy $(Y^{\uparrow\epsilon}, Y^{\downarrow\epsilon})$ and will converge to $\mathbb{E} \left[k^\uparrow \int_{\tau_{k-1}^\delta}^{\tau_k^\delta} e^{-\gamma t} dY^{\uparrow*}(t) \right] + \mathbb{E} \left[k^\downarrow \int_{\tau_{k-1}^\delta}^{\tau_k^\delta} e^{-\gamma t} dY^{\downarrow*}(t) \right]$. The first term in (A.16) can be written as follows for some $(u_1(\omega), u_2(\omega)) \in [0, \frac{\epsilon}{2}] \times [0, \frac{\epsilon}{2}]$, which is also a discrete Riemann sum of an integral

$$\begin{aligned} & \mathbb{E} \left[\sum_{j=1}^{j_k} e^{-\gamma t_{k,j}} \left(\frac{\partial C \left(\mathcal{X}_{t_{k,j-}}^{\delta,\epsilon}, u_1(\omega), u_2(\omega) \right)}{\partial \xi^\uparrow} \Delta_{k,j}^\uparrow + \frac{\partial C \left(\mathcal{X}_{t_{k,j-}}^{\delta,\epsilon}, u_1(\omega), u_2(\omega) \right)}{\partial \xi^\downarrow} \Delta_{k,j}^\downarrow \right) \right] \\ & \rightarrow \mathbb{E} \left[\int_{\tau_{k-1}^\delta}^{\tau_k^\delta} e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^\uparrow} dY^{\uparrow*}(t) + \int_{\tau_{k-1}^\delta}^{\tau_k^\delta} e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^\downarrow} dY^{\downarrow*}(t) \right] \end{aligned}$$

because $\max_{j=1,2,\dots,j_k} \Delta_{k,j} \rightarrow 0$ as $\epsilon \rightarrow 0$. Letting $\epsilon \rightarrow 0$, each term in (A.12) is greater than

$$\begin{aligned} & -2M\delta + \mathbb{E} \left[\int_{\tau_{k-1}^\delta}^{\tau_k^\delta} e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^\uparrow} dY^{\uparrow*}(t) + \int_{\tau_{k-1}^\delta}^{\tau_k^\delta} e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^\downarrow} dY^{\downarrow*}(t) \right] \\ & + \mathbb{E} \left[\int_{\tau_{k-1}^\delta}^{\tau_k^\delta} e^{-\gamma u} h(\mathcal{X}_u^\delta(0)) du \right] + \mathbb{E} \left[k^\uparrow \int_{\tau_{k-1}^\delta}^{\tau_k^\delta} e^{-\gamma t} dY^{\uparrow*}(t) \right] + \mathbb{E} \left[k^\downarrow \int_{\tau_{k-1}^\delta}^{\tau_k^\delta} e^{-\gamma t} dY^{\downarrow*}(t) \right]. \end{aligned} \tag{A.17}$$

Following the same argument, (A.13) is greater than

$$\begin{aligned} & -2M\delta + \mathbb{E} \left[\int_{\tau_{N(T)}^\delta}^T e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^\uparrow} dY^{\uparrow*}(t) + \int_{\tau_{N(T)}^\delta}^T e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^\downarrow} dY^{\downarrow*}(t) \right] \\ & + \mathbb{E} \left[\int_{\tau_{N(T)}^\delta}^T e^{-\gamma u} h(\mathcal{X}_u^\delta(0)) du \right] + \mathbb{E} \left[k^\uparrow \int_{\tau_{N(T)}^\delta}^T e^{-\gamma t} dY^{\uparrow*}(t) \right] + \mathbb{E} \left[k^\downarrow \int_{\tau_{N(T)}^\delta}^T e^{-\gamma t} dY^{\downarrow*}(t) \right]. \end{aligned} \tag{A.18}$$

Plugging (A.17) and (A.18) into (A.12) and (A.13), we have that

$$\begin{aligned} & C^*(\mathcal{X}_0^\delta) - \mathbb{E} \left[e^{-\gamma T} C^*(\mathcal{X}_T^\delta) \right] - \mathbb{E} [2N(T) + 2]M\delta \\ & \geq \mathbb{E} \left[\int_0^T e^{-\gamma u} h(\mathcal{X}_u^\delta(0)) du \right] + k^\uparrow \mathbb{E} \left[\int_0^T e^{-\gamma t} dY^{\uparrow*}(t) \right] + k^\downarrow \mathbb{E} \left[\int_0^T e^{-\gamma t} dY^{\downarrow*}(t) \right] \\ & + \mathbb{E} \left[\int_0^T e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^\uparrow} dY^{\uparrow*}(t) + \int_0^T e^{-\gamma t} \frac{\partial C(\mathcal{X}_t^\delta, 0, 0)}{\partial \xi^\downarrow} dY^{\downarrow*}(t) \right]. \end{aligned}$$

Combining the above with the cost function $C^\delta(\mathcal{X}, \pi^*)$ defined in (39), we have (A.10). \square

Proof of Proposition 11. It follows as

$$\begin{aligned}
 \mathbb{E} \left[\int_{\ell^1}^{\infty} e^{-\gamma t} h(H_t) dt \right] &= \mathbb{E} \left[\int_0^{\infty} e^{-\gamma(t+\ell^1)} h(H_{t+\ell^1}) dt \right] \\
 &= \mathbb{E} \left[\int_0^{\infty} e^{-\gamma(t+\ell^1)} h(H_0 + W_{t+\ell^1} + \mathcal{X}_0^\uparrow(t + \ell^1) - \mathcal{X}_0^\downarrow(t + \ell^1) + Y^\uparrow(t + \ell^1 - \ell^1) - Y^\downarrow(t)) dt \right] \\
 &= \mathbb{E} \left[\int_0^{\infty} e^{-\gamma(t+\ell^1)} h(W_{t+\ell^1} - W_t + W_t + H_0 + \mathcal{X}_0^\uparrow(t + \ell^1) - \mathcal{X}_0^\downarrow(\ell^1) + Y^\uparrow(t - \ell^1) - Y^\downarrow(t)) dt \right] \\
 &= \mathbb{E} \left[\int_0^{\infty} e^{-\gamma(t+\ell^1)} h(W_{t+\ell^1} - W_t + \mathcal{X}_0(t) + Y^\uparrow(t - \ell^1) - Y^\downarrow(t)) dt \right] \\
 &= \mathbb{E} \left[\int_0^{\infty} e^{-\gamma(t+\ell^1)} \mathbb{E} [h(W_{t+\ell^1} - W_t + \mathcal{X}_t(0)) | \mathcal{X}_t(0)] dt \right] \\
 &= \mathbb{E} \left[\int_0^{\infty} e^{-\gamma t} \mathbb{E} [e^{-\gamma \ell^1} h(\mathcal{X}_t(0) + \mathcal{N}_{\ell^1}) dt] \right] \\
 &= \mathbb{E} \left[\int_0^{\infty} e^{-\gamma t} \tilde{h}(\mathcal{X}_t(0)) dt \right].
 \end{aligned}$$

The cost difference $\mathbb{E} \left[\int_0^{\ell^1} e^{-\gamma t} h(H_t) dt \right]$ is a constant because, for $t \leq \ell^1$,

$$H_t = H_0 + W_t + \mathcal{X}_0^\uparrow(t) - \mathcal{X}_0^\downarrow(t) + Y^\uparrow(t - \ell^1) + Y^\downarrow(t - \ell^1) = H_0 + W_t + \mathcal{X}_0^\uparrow(t) - \mathcal{X}_0^\downarrow(t). \quad \square$$

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